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Wigner – Eckart

Zare / Edmonds

$$\langle n_1 j_1 m_1 | \hat{T}^{(J)}_M | n_2 j_2 m_2 \rangle \equiv (-1)^{2J} \frac{\langle n_1 j_1 \| \hat{T}^{(J)} \| n_2 j_2 \rangle}{[j_1]^{1/2}} \langle j_2 m_2 J M | j_1 m_1 \rangle \qquad [1]$$

$$= (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & J & j_2 \\ -m_1 & M & m_2 \end{pmatrix} \langle n_1 j_1 \| \hat{T}^{(J)} \| n_2 j_2 \rangle \qquad [2]$$

$$\hat{\tau}_{m_1 m_2}(j_1 j_2) \equiv |j_1 m_1\rangle \langle j_2 m_2| \qquad [3]$$

$$\hat{T}_{KQ}(j_1 j_2) \equiv \sum_{m_1 m_2} \hat{\tau}_{m_1 m_2}(j_1 j_2)\, (-1)^{2K + j_2 - m_2} \langle j_1 m_1\, j_2\, {-m_2} | K Q \rangle \qquad [4]$$

$$= \sum_{m_1 m_2} \hat{\tau}_{m_1 m_2}(j_1 j_2)\, (-1)^{j_1 - m_1} [K]^{1/2} \begin{pmatrix} j_1 & K & j_2 \\ -m_1 & Q & m_2 \end{pmatrix} \qquad [5]$$

$$\hat{T}^{(J)}_M = \sum_{n_1 n_2 j_1 m_1 j_2 m_2} |n_1 j_1 m_1\rangle \langle n_1 j_1 m_1 | \hat{T}^{(J)}_M | n_2 j_2 m_2 \rangle \langle n_2 j_2 m_2| \qquad [6]$$

$$\hat{T}^{(J)}_M = \sum_{j_1 j_2} \hat{T}_{JM}(j_1 j_2) \sum_{n_1 n_2} \frac{|n_1\rangle \langle n_1 j_1 \| \hat{T}^{(J)} \| n_2 j_2 \rangle \langle n_2|}{[J]^{1/2}} \qquad [7]$$

$$\hat{T}_{K_1 Q_1}(j_1 j_2)\, \hat{T}_{K_2 Q_2}(j_2 j_3) = \sum_{K_3 Q_3} \hat{T}_{K_3 Q_3}(j_1 j_3) [K_1]^{1/2} [K_2]^{1/2} (-1)^{j_1 + j_3 + K_3} \qquad [8]$$

$$\times \begin{Bmatrix} j_1 & j_2 & K_1 \\ K_2 & K_3 & j_3 \end{Bmatrix} \langle K_1 Q_1 K_2 Q_2 | K_3 Q_3 \rangle \qquad [9]$$

$$\hat{T}_{K_1 Q_1}(j_1 j_2)\, \hat{T}_{K_2 Q_2}(j_2 j_3)\, \hat{T}_{K_3 Q_3}(j_3 j_4) = \sum_{K_4 Q_4} \hat{T}_{K_4 Q_4}(j_1 j_4) [K_1]^{1/2} [K_2]^{1/2} [K_3]^{1/2} [K_4]^{1/2}$$

$$\times (-1)^{-j_3 + j_4 + K_2 + K_3 + Q_2} \sum_j [j] \begin{Bmatrix} j_1 & j_2 & K_1 \\ j_4 & j_3 & K_3 \\ K_4 & K_2 & j \end{Bmatrix} \begin{pmatrix} K_1 & j & K_3 \\ Q_1 & m & Q_3 \end{pmatrix} \begin{pmatrix} K_2 & j & K_4 \\ Q_2 & -m & -Q_4 \end{pmatrix} \qquad [10]$$

$(m = -Q_1 + Q_3)$

$$\hat{T}_{K_1 Q_1}(j_1 j_2)\, \hat{T}_{K_2 Q_2}(j_2 j_3)\, \hat{T}_{K_3 Q_3}(j_3 j_4)\, \hat{T}_{K_4 Q_4}(j_4 j_5) = \sum_{K_5 Q_5} \hat{T}_{K_5 Q_5}(j_1 j_5) [K_1]^{1/2} [K_2]^{1/2} [K_3]^{1/2} [K_4]^{1/2} [K_5]^{1/2}$$

$$\times (-1)^{-j_2 - j_5 + j_6 + K_1 - K_2 + K_3} \sum_{j_6 j_7} [j_6][j_7] \begin{Bmatrix} j_1 & j_2 & K_1 \\ j_5 & j_4 & K_4 \\ K_5 & j_7 & j_6 \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & K_2 \\ K_3 & j_7 & j_4 \end{Bmatrix}$$

$$\times \begin{pmatrix} K_1 & K_4 & j_6 \\ Q_1 & Q_4 & m_6 \end{pmatrix} \begin{pmatrix} K_2 & K_3 & j_7 \\ Q_2 & Q_3 & m_7 \end{pmatrix} \begin{pmatrix} j_6 & j_7 & K_5 \\ m_6 & m_7 & Q_5 \end{pmatrix} \qquad [11]$$

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Special cases of

$$\left[\hat{T}_{K_1}(j_1 j_2)\otimes\hat{T}_{K_2}(j_3 j_4)\right]^{(K_3)}_{Q_3}=\sum_{\mathcal{J}_1\mathcal{J}_2}\hat{T}_{K_3 Q_3}\left[(j_1 j_3)\mathcal{J}_1;(j_2 j_4)\mathcal{J}_2\right]\begin{Bmatrix} j_1 & j_2 & K_1\\ j_3 & j_4 & K_2\\ \mathcal{J}_1 & \mathcal{J}_2 & K_3\end{Bmatrix}[\mathcal{J}_1]^{1/2}[\mathcal{J}_2]^{1/2}[K_1]^{1/2}[K_2]^{1/2}$$

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(3) $K_3=0 \Rightarrow K_1=K_2\ ;\ \mathcal{J}_1=\mathcal{J}_2$

$$\begin{Bmatrix} j_1 & j_2 & K_1\\ j_3 & j_4 & K_1\\ \mathcal{J}_1 & \mathcal{J}_1 & 0\end{Bmatrix}=\frac{(-1)^{j_2+K_1+j_3+\mathcal{J}_1}}{[K_1]^{1/2}[\mathcal{J}_1]^{1/2}}\begin{Bmatrix} j_1 & j_2 & K_1\\ j_4 & j_3 & \mathcal{J}_1\end{Bmatrix}$$

$$\left[\hat{T}_{K_1}(j_1 j_2)\otimes\hat{T}_{K_1}(j_3 j_4)\right]^{(0)}_{0}=\sum_{\mathcal{J}_1}\hat{T}_{00}\left[(j_1 j_3)\mathcal{J}_1;(j_2 j_4)\mathcal{J}_1\right]\begin{Bmatrix} j_1 & j_2 & K_1\\ j_4 & j_3 & \mathcal{J}_1\end{Bmatrix}[\mathcal{J}_1]^{1/2}[K_1]^{1/2}(-1)^{j_2+j_3+K_1+\mathcal{J}_1}$$

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(1) $K_1=0 \Rightarrow j_1=j_2\ ,\ K_2=K_3\quad Q_3=Q_2$ $\qquad \langle 0\,0\,K_2 Q_2|K_3 Q_3\rangle=\delta_{K_2K_3}\delta_{Q_2Q_3}$

$$\begin{Bmatrix} j_1 & j_1 & 0\\ j_3 & j_4 & K_2\\ \mathcal{J}_1 & \mathcal{J}_2 & K_2\end{Bmatrix}=\frac{(-1)^{\mathcal{J}_1+K_2+j_4+j_1}}{[K_2]^{1/2}[j_1]^{1/2}}\begin{Bmatrix} \mathcal{J}_2 & \mathcal{J}_1 & K_2\\ j_3 & j_4 & j_1\end{Bmatrix}$$

$$[j_1]^{1/2}\hat{T}_{00}(j_1 j_1)\otimes\hat{T}_{K_2Q_2}(j_3 j_4)=\sum_{\mathcal{J}_1\mathcal{J}_2}\hat{T}_{K_2Q_2}\left[(j_1 j_3)\mathcal{J}_1;(j_1 j_4)\mathcal{J}_2\right]\begin{Bmatrix} \mathcal{J}_2 & \mathcal{J}_1 & K_2\\ j_3 & j_4 & j_1\end{Bmatrix}[\mathcal{J}_1]^{1/2}[\mathcal{J}_2]^{1/2}\cancel{[j_1]^{-1/2}}\times(-1)^{j_1+j_4+K_2+\mathcal{J}_1}$$

$K_2=0 \Rightarrow j_3=j_4\ ;\ K_3=K_1$

$$\begin{Bmatrix} j_1 & j_2 & K_1\\ j_3 & j_3 & 0\\ \mathcal{J}_1 & \mathcal{J}_2 & K_1\end{Bmatrix}=\frac{(-1)^{j_1+K_1+\mathcal{J}_2+j_3}}{[K_1]^{1/2}[j_3]^{1/2}}\begin{Bmatrix} j_2 & j_1 & K_1\\ \mathcal{J}_1 & \mathcal{J}_2 & j_3\end{Bmatrix}$$

$$\hat{T}_{K_1Q_1}(j_1 j_2)\otimes\hat{T}_{00}(j_3 j_3)[j_3]^{1/2}=\sum_{\mathcal{J}_1\mathcal{J}_2}\hat{T}_{K_1Q_1}\left[(j_1 j_3)\mathcal{J}_1;(j_2 j_3)\mathcal{J}_2\right]\begin{Bmatrix} j_2 & j_1 & K_1\\ \mathcal{J}_1 & \mathcal{J}_2 & j_3\end{Bmatrix}[\mathcal{J}_1]^{1/2}[\mathcal{J}_2]^{1/2}\times(-1)^{j_1+j_3+K_1+\mathcal{J}_2}$$

$$\boxed{[j_1]^{1/2}\hat{T}_{0,0}(j_1 j_1)\otimes[j_2]^{1/2}\hat{T}_{0,0}(j_2 j_2)=\sum_{\mathcal{J}}[\mathcal{J}]^{1/2}\hat{T}_{0,0}\left[(j_1 j_2)\mathcal{J};(j_1 j_2)\mathcal{J}\right]}$$

$$\hat{I}=\sum_j [j]^{1/2}\hat{T}_{00}(j,j)$$ [Note: unpolarized state $\hat{\rho}=[j]^{-1/2}\hat{T}_{00}(j,j)$]

B.F.

$$\hat{T}_{K_1Q_1}(j_1 j_2)\otimes\hat{I}=\sum_{\mathcal{J}_1\mathcal{J}_2\Omega_1\Omega_2}\hat{T}_{K_1Q_1}\left[j_1\Omega_1\mathcal{J}_1;j_2\Omega_2\mathcal{J}_2\right][\mathcal{J}_1]^{1/2}[\mathcal{J}_2]^{1/2}\begin{pmatrix} j_1 & K_1 & j_2\\ \bar{\Omega}_1 & q & \Omega_2\end{pmatrix}\begin{pmatrix} \mathcal{J}_1 & K_1 & \mathcal{J}_2\\ \bar{\Omega}_1 & q & \Omega_2\end{pmatrix}$$

$q\equiv\Omega_1-\Omega_2$

$$\hat{A}^{SF}_{K_1Q_1}=\sum_{j_1j_2}\hat{T}_{K_1Q_1}(j_1 j_2)\,a(j_1 j_2)$$

$$=\sum_{\mathcal{J}_1\mathcal{J}_2 j_1 j_2\Omega_1\Omega_2}\hat{T}_{K_1Q_1}\left[j_1\Omega_1\mathcal{J}_1;j_2\Omega_2\mathcal{J}_2\right]\langle j_1\Omega_1|\hat{A}^{BF}_{K_1q}|j_2\Omega_2\rangle\begin{pmatrix} \mathcal{J}_1 & K_1 & \mathcal{J}_2\\ \bar{\Omega}_1 & q & \Omega_2\end{pmatrix}\frac{[\mathcal{J}_1]^{1/2}[\mathcal{J}_2]^{1/2}}{[K_1]^{1/2}}$$

Alternative

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$$\hat{T}_{K_1Q_1}(j_1j_2)\ \hat{T}_{K_2Q_2}(j_2j_3)\ \hat{T}_{K_3Q_3}(j_3j_4)\ \hat{T}_{K_4Q_4}(j_4j_5)$$

$$= \sum_{k_a q_a} \hat{T}_{K_AQ_A}(j_1j_3)\,[K_1]^{1/2}[K_2]^{1/2}(-1)^{j_1+j_3+K_A} \begin{Bmatrix} j_1 & j_2 & K_1 \\ K_2 & K_A & j_3 \end{Bmatrix} \langle K_1Q_1\,K_2Q_2 | K_AQ_A\rangle$$

$$\sum_{K_BQ_B} \hat{T}_{K_BQ_B}(j_3j_5)\,[K_3]^{1/2}[K_4]^{1/2}(-1)^{j_3+j_5+K_B} \begin{Bmatrix} j_3 & j_4 & K_3 \\ K_4 & K_B & j_5 \end{Bmatrix} \langle K_3Q_3\,K_4Q_4 | K_BQ_B\rangle$$

$$= \sum_{KQ} \hat{T}_{KQ}(j_1j_5)\,[K_1]^{1/2}[K_2]^{1/2}[K_3]^{1/2}[K_4]^{1/2}[K_A]^{1/2}[K_B]^{1/2}$$

$$K_AK_B \quad (-1)^{j_1+j_3+K_A+j_3+j_5+K_B+j_1+j_5+K}$$

$$\begin{Bmatrix} j_1 & j_2 & K_1 \\ K_2 & K_A & j_3 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & K_3 \\ K_4 & K_B & j_5 \end{Bmatrix} \begin{Bmatrix} j_1 & j_3 & K_A \\ K_B & K & j_5 \end{Bmatrix}$$

$$\langle K_1Q_1\,K_2Q_2 | K_AQ_A\rangle \langle K_3Q_3\,K_4Q_4 | K_BQ_B\rangle \langle K_AQ_A\,K_BQ_B | KQ\rangle$$

phase $(-1)^{2j_1+2j_3+2j_5+K_A+K_B+K}$

$$\hat{T}^{\dagger}_{KQ}(j_1j_2) = (-1)^{j_2-j_1+Q}\,\hat{T}_{K,-Q}(j_2j_1)$$

Summary

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(1)

$$T_{k_1 q_1}(j_1 j_2) = \sum_{m_1 m_2} |j_1 m_1\rangle\langle j_2 m_2| \, \langle j_1 m_1, j_2 \bar m_2 | k_1 q_1\rangle (-1)^{2k_1}$$

$$T_{k_2 q_2}(j_3 j_4) = \sum_{m_3 m_4} |j_3 m_3\rangle\langle j_4 m_4| \, \langle j_3 m_3 \, j_4 \bar m_4 | k_2 q_2\rangle (-1)^{2k_2}$$

$$|(j_1 j_3) J_1 M_1\rangle = \sum_{m_1 m_3} |j_1 m_1\rangle |j_3 m_3\rangle \langle j_1 m_1 \, j_3 m_3 | J_1 M_1\rangle$$

$$|(j_2 j_4) J_2 M_2\rangle = \sum_{m_2 m_4} |j_2 m_2\rangle |j_4 m_4\rangle \langle j_2 m_2 \, j_4 m_4 | J_2 M_2\rangle$$

$$T_{k_3 q_3}[(j_1 j_3) J_1, (j_2 j_4) J_2] = \sum |(j_1 j_3) J_1 M_1\rangle \langle (j_2 j_4) J_2 M_2| \, \langle J_1 M_1 \, J_2 \bar M_2 | k_3 q_3\rangle (-1)^{2k_3}$$

$$T_{k_1 q_1}(j_1 j_2) \, T_{k_2 q_2}(j_3 j_4) = \sum_{J_1 J_2 k_3 q_3} T_{k_3 q_3}[(j_1 j_3) J_1; (j_2 j_4) J_2] \; C_{k_3 q_3}(j_1 j_2 J_1; j_3 j_4 J_2)$$

$$C_{k_3 q_3}(j_1 j_2 J_1, j_3 j_4 J_2) = \mathrm{Tr}\left\{ T^\dagger_{k_3 q_3}[(j_1 j_3) J_1, (j_2 j_4) J_2] \, [T_{k_1 q_1}(j_1 j_2) \otimes T_{k_2 q_2}(j_3 j_4)] \right\}$$

$$= \begin{Bmatrix} j_1 & j_2 & k_1 \\ j_3 & j_4 & k_2 \\ J_1 & J_2 & k_3 \end{Bmatrix} \langle k_1 q_1 \, k_2 q_2 | k_3 q_3\rangle \, [J_1]^{1/2} [J_2]^{1/2} [k_1]^{1/2} [k_2]^{1/2}$$

$$\left[T_{k_1}(j_1 j_2) \otimes T_{k_2}(j_3 j_4) \right]^{(k_3)}_{q_3} = \sum T_{k_3 q_3}[(j_1 j_3) J_1, (j_2 j_4) J_2] \begin{Bmatrix} j_1 & j_2 & k_1 \\ j_3 & j_4 & k_2 \\ J_1 & J_2 & k_3 \end{Bmatrix} [J_1]^{1/2} [J_2]^{1/2} \times [k_1]^{1/2} [k_2]^{1/2}$$

[B&L 3.24.8]

[B&S 5.12]

$$|j \bar m\rangle = (-1)^{j-m} |j \, -m\rangle$$

$$\langle j_1 m_1 \, j_2 m_2 | j_3 m_3\rangle = (-1)^{j_1 - j_2 + m_3} [j_3]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & \bar m_3 \end{pmatrix}$$

$$= (-1)^{2j_1} \qquad = (-1)^{2j_2}$$

$$T_{kq}(j_1 j_2) = \sum_{m_1 m_2} |j_1 m_1\rangle\langle j_2 m_2| \, \langle j_1 m_1 \, j_2 \bar m_2 | kq\rangle (-1)^{2j_1}$$

$$= \sum_{m_1 m_2} |j_1 m_1\rangle\langle j_2 m_2| \, \langle j_1 m_1 \, j_2 \, -m_2 | kq\rangle (-1)^{j_2 - m_2 + 2k}$$

$$= \sum_{m_1 m_2} |j_1 m_1\rangle\langle j_2 m_2| \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix} (-1)^{j_1 - m_1} [k]^{1/2}$$

$$= \sum_{m_1 m_2} |j_1 m_1\rangle\langle j_2 m_2| \, \langle j_2 m_2 \, k q | j_1 m_1\rangle (-1)^{2k}$$