Question 1: Angular momentum operators

The angular momentum operators $\hat{\boldsymbol{l}}$ for a single particle with position $\boldsymbol{r}$ and momentum $\hat{\boldsymbol{p}} = \frac{\hbar}{i}\boldsymbol{\nabla}$ are given by

$$\hat{\boldsymbol{l}} = \boldsymbol{r} \times \hat{\boldsymbol{p}}.$$

The position $\boldsymbol{r}$ expressed in spherical polar coordinates (r, θ, ϕ) is given by

$$\boldsymbol{r} = r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.$$

1a. Show that $\hat{l}_z$ in spherical polar coordinates is given by

$$\hat{l}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

Question 2: Rotations in $\mathbb{R}^3$

A rotation in $\mathbb{R}^3$ around a vector $\boldsymbol{n}$ with $|\boldsymbol{n}| = 1$ over an angle ϕ is given by

$$\boldsymbol{R}(\boldsymbol{n}, \phi) = e^{\phi\boldsymbol{N}},$$

where $\boldsymbol{N}$ is an anti-Hermitian matrix, implicitly defined by

$$\boldsymbol{n} \times \boldsymbol{x} = \boldsymbol{N}\boldsymbol{x}$$

with

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$$

2a. Find the matrices $\boldsymbol{N}_1$, $\boldsymbol{N}_2$, and $\boldsymbol{N}_3$ such that

$$\boldsymbol{N} = n_1\boldsymbol{N}_1 + n_2\boldsymbol{N}_2 + n_3\boldsymbol{N}_3$$

2b. Show that

$$\boldsymbol{N}^T = -\boldsymbol{N}.$$

2c. Derive the commutation relations

$$[\boldsymbol{N}_1, \boldsymbol{N}_2] = \boldsymbol{N}_3.$$

Question 3: Wigner rotation matrices

A rotation operator acting on the $(2l+1)$ dimensional linear space

$$\{|lm\rangle,\ m=-l,-l+1,\ldots,l\}$$

is given by

$$\hat{R}(\boldsymbol{n},\phi)=e^{-\frac{i}{\hbar}\phi\,\boldsymbol{n}\cdot\hat{\boldsymbol{l}}}.$$

Wigner rotation matrices $\boldsymbol{D}^{(l)}(\boldsymbol{n},\phi)$ are defined by

$$\hat{R}(\boldsymbol{n},\phi)|lm\rangle=\sum_{m'=-l}^{l}|lm'\rangle D^{(l)}_{m',m}(\boldsymbol{n},\phi).$$

We will use the short-hand notation $\boldsymbol{D}^{(l)}(\hat{R})\equiv\boldsymbol{D}^{(l)}(\boldsymbol{n},\phi)$.

3a. Show that $\boldsymbol{D}^{(l)}$ is a *representation* of $\hat{R}$, i.e.,

$$\boldsymbol{D}^{(l)}(\hat{R}_1\hat{R}_2)=\boldsymbol{D}^{(l)}(\hat{R}_1)\boldsymbol{D}^{(l)}(\hat{R}_2).$$

3b. Show that rotation over $\phi=0$ is represented by the $(2l+1)\times(2l+1)$ identity matrix:

$$\boldsymbol{D}^{(l)}(\boldsymbol{n},0)=\boldsymbol{I}_{(2l+1)\times(2l+1)}.$$

3c. Use the representation property to show that

$$\boldsymbol{D}^{(l)}(\hat{R}^\dagger)=\boldsymbol{D}^{(l)}(\hat{R})^\dagger.$$

Question 4: Euler angles

A rotation may be expressed in zyz Euler angles by

$$\hat{R}(\alpha,\beta,\gamma)=\hat{R}(\boldsymbol{e}_z,\alpha)\hat{R}(\boldsymbol{e}_y,\beta)\hat{R}(\boldsymbol{e}_z,\gamma). \tag{1}$$

4a. Show that

$$D^{(l)}_{m,k}(\alpha,\beta,\gamma)\equiv\langle lm|\hat{R}(\alpha,\beta,\gamma)|lk\rangle=e^{-im\alpha}d^{(l)}_{m,k}(\beta))e^{-ik\gamma}$$

with

$$\boldsymbol{d}^{(l)}(\beta)\equiv\boldsymbol{D}^{(l)}(\boldsymbol{e}_y,\beta).$$

Note: the matrix $\boldsymbol{d}^{(l)}(\beta)$ is real.

Question 5: Spherical harmonic addition theorem

Two normalized vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{k}}$ are defined by rotations of $\boldsymbol{e}_z$,

$$\begin{aligned} \hat{\boldsymbol{r}} &= \boldsymbol{R}_1 \boldsymbol{e}_z, \\ \hat{\boldsymbol{k}} &= \boldsymbol{R}_2 \boldsymbol{e}_z. \end{aligned}$$

The angle between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{k}}$ is θ,

$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{k}} = \cos\theta.$$

The scalar product can be written as

$$\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{k}} = (\boldsymbol{R}_1 \boldsymbol{e}_z) \cdot (\boldsymbol{R}_2 \boldsymbol{e}_z) = \boldsymbol{e}_z \cdot \boldsymbol{R}_1^\dagger \boldsymbol{R}_2 \boldsymbol{e}_z.$$

The rotation $\boldsymbol{R}_1^\dagger \boldsymbol{R}_2$ can be expressed in zyz Euler angles (α, β, γ)

$$\boldsymbol{R}_1^\dagger \boldsymbol{R}_2 = \boldsymbol{R}(\alpha, \beta, \gamma) = \boldsymbol{R}(\boldsymbol{e}_z, \alpha) \boldsymbol{R}(\boldsymbol{e}_y, \beta) \boldsymbol{R}(\boldsymbol{e}_z, \gamma)$$

5a. Show that

$$\boldsymbol{e}_z \cdot \boldsymbol{R}(\alpha, \beta, \gamma) \boldsymbol{e}_z = \cos\beta$$

5b. Show that $\cos\beta = \cos\theta$.

We now have established that

$$P_l(\cos\theta) \equiv d^{(l)}_{0,0}(\theta) = \boldsymbol{D}^{(l)}_{0,0}(\boldsymbol{R}_1^\dagger \boldsymbol{R}_2).$$

Spherical harmonics Y_{lm}, Racah normalized spherical harmonics C_{lm}, and Legendre polynomials P_l may be expressed as special cases of Wigner rotations matrices by

$$\begin{aligned} C_{lm}(\theta, \phi) &\equiv D^{(l)}_{m,0}(\phi, \theta, 0)^* \\ P_l(\cos\theta) &\equiv C_{l,0}(\theta, 0) \\ Y_{lm}(\theta, \phi) &\equiv \sqrt{\frac{2l+1}{4\pi}} C_{lm}(\theta, \phi). \end{aligned}$$

5c. Derive the spherical harmonics addition theorem

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-1}^{l} Y_{lm}(\hat{\boldsymbol{r}}) Y_{lm}(\hat{\boldsymbol{k}})^*.$$