Quantum Dynamics, NWI-SM295, exercises week 7

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Question 1: Inelastic scattering

The boundary conditions for inelastic scattering are given by

$$\Psi_n(\boldsymbol{r}) \cong |n\rangle v_n^{-\frac{1}{2}} e^{i\boldsymbol{k}_n\cdot\boldsymbol{r}} + \sum_{n'} |n'\rangle v_{n'}^{-\frac{1}{2}} \frac{e^{ik_{n'}r}}{r} f_{n'\leftarrow n}(\hat{r},\hat{k})$$

where n stands for all quantum numbers of the collision partners, $\boldsymbol{k}_n$ is the wave vector and $\boldsymbol{r}$ the vector connecting the centers-of-mass, f is the scattering amplitude, $k = |\boldsymbol{k}|$, $\boldsymbol{k} = k\hat{k}$, $r = |\boldsymbol{r}|$, and $\boldsymbol{r} = r\hat{r}$, and v_n is the velocity.

Using the partial wave expansion of the plane wave

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \cong \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\hat{k}\cdot\hat{r}).$$

and the spherical harmonic addition theorem

$$P_l(\hat{k}\cdot\hat{r}) = \frac{4\pi}{2l+1} \sum_{m_l=-l}^{l} Y_{lm_l}(\hat{r}) Y^*_{lm_l}(\hat{k})$$

the boundary conditions can be written in $S-$matrix form

$$\Psi_n(\boldsymbol{r}) \cong \frac{2\pi}{ik_n r} \sum_{n'} \sum_{l'm_l'} \sum_{lm_l} |n'\rangle Y_{l'm_l'}(\hat{r}) v_{n'}^{-\frac{1}{2}} \left[-e^{-i(k_{n'}r - \frac{1}{2}l'\pi)} \delta_{n'n}\delta_{l'l}\delta_{m_l',m_l} + e^{i(k_{n'}r-\frac{1}{2}l'\pi)} S_{n'l'm_l';nlm_l} \right] i^l Y_{lm_l}(\hat{k})^*.$$

The S-matrix is related to the T-matrix by

$$S_{n'l'm_l';nlm_l} = \delta_{n'n}\delta_{l'l}\delta_{m_l'm_l} - T_{n'l'm_l';nlm_l}. \tag{1}$$

1a. Derive an expression of the scattering amplitude $f_{n'\leftarrow n}(\hat{r},\hat{k})$ in terms of T-matrix elements.

The state-to-state differential cross section is given by

$$\frac{d\sigma_{n'\leftarrow n}}{d\Omega}(\hat{r},\hat{k}) = \frac{j_{\text{out}}}{j_{\text{in}}} = |f_{n'\leftarrow n}(\hat{r},\hat{k})|^2 \tag{2}$$

and the state-to-state integral cross section is given by

$$\sigma_{n'\leftarrow n} = \iint \frac{d\sigma_{n'\leftarrow n}}{d\Omega}(\hat{r},\hat{k}) d\hat{r}. \tag{3}$$

1b. Derive and expression for the integral cross section. Use the orthonormality of spherical harmonics Y_{lm}. Also, take the incoming direction to be the z-axis $\hat{\boldsymbol{k}} = \boldsymbol{e}_z$ and use

$$Y_{lm}(\boldsymbol{e}_z) \equiv Y_{lm}(0,0) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m,0}. \tag{4}$$