Quantum Dynamics, NWI-SM295, exercises week 6

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Question 1: Elastic scattering in 3D

A plane wave traveling along the z-axis can be expanded in spherical waves:

$$e^{ikrz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(z), \tag{1}$$

where $z \equiv \cos\theta$, the position of the particle is given by the spherical coordinates (r, θ, ϕ), $j_l(kr)$ are spherical Bessel functions of the first kind, $P_l(z)$ are Legendre polynomials of order l, and the wave vector is $\boldsymbol{k} = k\boldsymbol{e}_z$.

The Legendre polynomials are defined by

$$P_0(z) = 1 \tag{2}$$

$$P_1(z) = z \tag{3}$$

$$P_{l+1}(z) = \frac{z(2l+1)P_l(z) - lP_{l-1}(z)}{l+1} \tag{4}$$

and they satisfy the orthogonality relations

$$\int_{-1}^{1} P_l(z) P_{l'}(z) dz = \frac{2}{2l+1} \delta_{ll'}.$$

1a. Use the orthogonality relation of the Legendre polynomials and the plane wave expansion to derive and explicit expression for $j_0(kr)$, i.e., project Eq. (1) onto $P_0(z)$.

1b. Also project Eq. (1) onto $P_1(z)$ to find the expression for $j_1(kr)$.

Equation 10.1.20 of A&S gives the following expression for the derivative of spherical Bessel function of the first kind:

$$(2l+1)\frac{\partial}{\partial x} j_l(x) = -(l+1) j_{l+1}(x) + l j_{l-1}(x). \tag{5}$$

1c. Show that this expression is consistent with the implicit definition of spherical Bessel functions in Eq. (1). Hint: define $x = kr$, then take the derivative of Eq.(1) with respect to x, project the equation by $P_l(z)$, and use the recursion relation of Legendre polynomials to rewrite $zP_l(z)$.

Question 2: Explicit expression for scattering amplitude

The boundary conditions for elastic scattering of a single particle are given by

$$\psi(\boldsymbol{r}) = e^{ikrz} + \frac{e^{ikr}}{r} f(\theta).$$

A nonzero potential will modify only the outgoing part of the plane wave [Eq. (1)]. For sufficiently large r the spherical Bessel functions of the first kind take the form

$$j_l(kr) = \frac{\sin(kr - l\frac{\pi}{2})}{kr}.$$

The sin function can be written as an incoming and an outgoing wave using

$$2i \sin x = -e^{-ix} + e^{ix}$$

and the effect of the potential is to modify the outgoing part:

$$-e^{-ix} + e^{ix} \Rightarrow -e^{-ix} + e^{ix} S.$$

The S may be different for each partial wave (each value of l), so the scattering amplitude $f(\theta)$ can be written as a sum involving "S-matrices" S_l.

2a. Derive the expression for the scattering amplitude $f(\theta)$ in terms of the S-matrixes S_l.

Question 3: Differential and integral cross sections

The differential cross section is related to the scattering amplitude

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2.$$

The integral cross section is obtained by integrating over all scattering angles

$$\sigma = \int_0^{2\pi} d\phi \int_{-1}^{1} |f(\theta)|^2 \, d\cos\theta.$$

3a. Use the expression for $f(\theta)$ from the previous question and the orthogonality of the Legendre polynomials to find and explicit expression for the integral cross sections in terms of the S_l's.