QTC, NWI-MOL112, exercises week 3

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Question 1: Questions chapter 5

In section 5.1 of the lecture notes, the angular momentum states $|ab\rangle$ are defined by

$$\hat{l}^2|ab\rangle = a\hbar^2|ab\rangle \tag{1}$$

$$\hat{l}_z|ab\rangle = b\hbar|ab\rangle. \tag{2}$$

Ladder operators are defined by

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y. \tag{3}$$

1a. Show that $b^2 \leq a$, without using any of well known relations for the l and m quantum numbers of angular momentum states.

1b. The angular momentum ladder operators are eachothers Hermitian conjugates, $\hat{l}_\pm^\dagger = \hat{l}_\mp$. Derive this result using the definition of Hermitian conjugate and the defining properties of scalar products.

1c. Show that

$$\hat{l}_\pm\hat{l}_\mp = \hat{l}^2 - \hat{l}_z^2 \pm \hbar\hat{l}_z.$$

The angular momentum operator $\hat{l}_z$ in spherical polar coordinates, is given by [see lecture notes Eq. (5.48)]

$$\hat{l}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{4}$$

1d. Derive this result starting from the expression for $\hat{l}_z$ in Cartesian coordinates.

1e. In the derivation in Chapter 5.8 we used

$$(\boldsymbol{n} \times \boldsymbol{r}) \cdot \boldsymbol{\nabla} = \boldsymbol{n} \cdot (\boldsymbol{r} \times \boldsymbol{\nabla})$$

Derive this equation using the Levi-Civita tensor.

1f. Compute the matrix elements of the rotation operator

$$\langle lm|\hat{R}(\boldsymbol{e}_z, \alpha)|lm'\rangle.$$

1g. For $l = 1/2$, the possible values of $m = -1/2, 1/2$. Denoting the angular momentum states by $|lm\rangle$, compute the matrix representation of $\hat{l}^2$, $\hat{l}_z$, $\hat{l}_\pm$, $\hat{l}_x$, and $\hat{l}_y$ in the basis $\{|1/2, -1/2\rangle, |1/2, 1/2\rangle\}$.

1h. Compute the Wigner D-matrix elements

$$d^{(l)}_{mk}(\beta) = \langle lm|e^{-\frac{i}{\hbar}\beta\hat{l}_y}|lk\rangle$$

for $l = 1/2$.

1i. Show that the Wigner-D matrices satisfy the matrix representation property

$$\boldsymbol{D}^{(l)}(\hat{R}_1\hat{R}_2) = \boldsymbol{D}^{(l)}(\hat{R}_1)\boldsymbol{D}^{(l)}(\hat{R}_2), \tag{5}$$

starting from the defining equation of the $\boldsymbol{D}$-matrices.