Question 1: Questions chapter 5

In section 5.1 of the lecture notes, the angular momentum states $|ab\rangle$ are defined by

$$\hat{l}^2|ab\rangle = \hbar^2|ab\rangle$$  \hspace{1cm} (1)
$$\hat{l}_z|ab\rangle = \hbar|ab\rangle.$$  \hspace{1cm} (2)

Ladder operators are defined by

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y.$$  \hspace{1cm} (3)

1a. Show that $b^2 \leq a$, without using any of well known relations for the $l$ and $m$ quantum numbers of angular momentum states.

**Answer:** The expectation value of the square of a Hermitian operator must be positive, e.g.,

$$\langle ab|\hat{l}^2|ab\rangle = \langle \hat{l}_x ab||\hat{l}_x ab \rangle = |\langle \hat{\hat{l}}_x |ab \rangle|^2 \geq 0.$$  \hspace{1cm} (4)

Since

$$\langle ab|\hat{l}^2|ab\rangle - \langle ab|\hat{l}^2_z|ab\rangle = \langle ab|\hat{l}^2 + \hat{l}^2_y|ab\rangle - \langle ab|\hat{l}^2_z|ab\rangle = \langle ab|\hat{l}^2_z|ab\rangle + \langle ab|\hat{l}^2_y|ab\rangle \geq 0$$  \hspace{1cm} (5)

$$a\hbar^2 - (b\hbar)^2 \geq 0$$  \hspace{1cm} (6)

$$a \geq b^2.$$  \hspace{1cm} (7)

1b. The angular momentum ladder operators are each other’s Hermitian conjugates, $\hat{l}^\dagger_\pm = \hat{l}^\pm$. Derive this result using the definition of Hermitian conjugate and the defining properties of scalar products.

**Answer:**

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y.$$  \hspace{1cm} (8)

Since $\hat{l}_x$ is Hermitian we have $\hat{l}^\dagger_x = \hat{l}_x$. Thus, we have to show

$$(i\hat{l}_y)^\dagger = -\hat{l}_y.$$  \hspace{1cm} (9)

Since $\hat{l}_y$ is Hermitian, for any state $s|\phi\rangle, \chi$ we have

$$\langle \phi|\hat{l}_y\chi\rangle = \langle \hat{l}_y\phi|\chi \rangle.$$  \hspace{1cm} (10)

Thus, for these states we also have,

$$\langle \phi|i\hat{l}_y\chi\rangle = i\langle \hat{l}_y\phi|\chi \rangle = \langle -i\hat{l}_y\phi|\chi \rangle,$$  \hspace{1cm} (11)

and hence $(i\hat{l}_y)^\dagger = -i\hat{l}_y$, q.e.d.

1c. Show that

$$\hat{l}_\pm\hat{l}_\mp = \hat{l}^2 - \hat{l}^2_z \pm \hbar\hat{l}_z.$$  \hspace{1cm} (12)

**Answer:**

$$\hat{l}_\pm\hat{l}_\mp = (\hat{l}_x \pm i\hat{l}_y)(\hat{l}_x \mp i\hat{l}_y)$$  \hspace{1cm} (13)
$$= \hat{l}_x^2 + \hat{l}_y^2 \mp i[\hat{l}_x, \hat{l}_y]$$  \hspace{1cm} (14)
$$= \hat{l}^2 - \hat{l}^2_z \mp i^2\hbar\hat{l}_z$$  \hspace{1cm} (15)
The angular momentum operator \( \hat{l}_z \) in spherical polar coordinates, is given by [see lecture notes Eq. (5.48)]

\[
\hat{l}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}.
\] (16)

1d. Derive this result starting from the expression for \( \hat{l}_z \) in Cartesian coordinates.

**Answer:** From the definition:

\[
\hat{l}_z = x \hat{p}_y - y \hat{p}_x = \frac{\hbar}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).
\] (17)

With

\[
x = r \cos \phi \sin \theta \] (18)
\[
y = r \sin \phi \sin \theta \] (19)
\[
z = r \cos \theta \] (20)

and starting from Eq. (16) we find

\[
\hat{l}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \] (21)
\[
= \frac{\hbar}{i} \left( r \cos \phi \sin \theta \frac{\partial}{\partial y} - r \sin \phi \sin \theta \frac{\partial}{\partial x} + 0 \right) \] (22)
\[
= \frac{\hbar}{i} \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right), \] (23)

q.e.d.

1e. In the derivation in Chapter 5.7 we used

\[
(n \times r) \cdot \nabla = n \cdot (r \times \nabla)
\]

Derive this equation using the Levi-Civita tensor.

**Answer:** On the lhs we have

\[
(n \times r) \cdot \nabla = \epsilon_{ijk} n_j r_k \nabla_i.
\] (25)

On the rhs we have

\[
n \cdot (r \times \nabla) = \epsilon_{ijk} n_i r_j \nabla_k
\] (26)

The Levi-Civita tensor is invariant under cyclic permutations, so \( \epsilon_{ijk} = \epsilon_{kij} \), so we have

\[
n \cdot (r \times \nabla) = \epsilon_{kji} n_j r_i \nabla_k = \epsilon_{ijk} n_j r_k \nabla_i = (n \times r) \cdot \nabla.
\] (27)

where we swapped the summation indices \( k \) and \( i \) in the second step.

1f. Compute the matrix elements of the rotation operator

\[
\langle lm | \hat{R}(e_z, \alpha) | l'm' \rangle.
\]

**Answer:**

\[
\langle lm | \hat{R}(e_z, \alpha) | l'm' \rangle = \langle lm | e^{-\gamma \alpha_+} | l'm' \rangle = e^{-im\alpha} \delta_{mm'}.
\] (28)

1g. For \( l = 1/2 \), the possible values of \( m = -1/2, 1/2 \). Denoting the angular momentum states by \( |lm \rangle \), compute the matrix representation of \( \hat{l}_z \), \( \hat{l}_z \), \( \hat{l}_\pm \), \( \hat{l}_x \), and \( \hat{l}_y \) in the basis \( \{|1/2, -1/2\}, |1/2, 1/2\} \).
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Answer: We have \( l = 1/2 \), so \( l(l+1) = 3/4 \), hence the matrix representation of \( \hat{l}^2 \) is

\[
\hat{L}^2 = \hbar^2 \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}
\] (29)

We have \( m = \mp \frac{1}{2} \), so the matrix representation of \( \hat{l}_z \) is

\[
\hat{L}_z = \hbar \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}
\] (30)

Note that the \(-\frac{1}{2}\) is in the first column, since the first basis function has \( m = -\frac{1}{2} \).

For the ladder operators we have

\[
\hat{l}_+ |lm\rangle = \hbar \sqrt{l(l+1) - m(m+1)} |l,m\rangle
\] (31)

\[
\hat{l}_- |\frac{1}{2}, -\frac{1}{2}\rangle = \hbar \sqrt{\frac{1}{2} \left( \frac{1}{2} + 1 \right) - \frac{1}{2} \left( \frac{1}{2} - 1 \right)} |\frac{1}{2}, \frac{1}{2}\rangle
\] (32)

\[
\hat{l}_- |\frac{1}{2}, \frac{1}{2}\rangle = \hbar \sqrt{\frac{1}{2} \left( \frac{1}{2} + 1 \right) - \frac{1}{2} \left( \frac{1}{2} - 1 \right)} |\frac{1}{2}, -\frac{1}{2}\rangle
\] (33)

In the matrix representation, the first column is represents the image of the first basis function, \( |\frac{1}{2}, -\frac{1}{2}\rangle \), etc., so

\[
\hat{L}_+ = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\] (34)

\[
\hat{L}_- = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\] (35)

Since \( \hat{l}_x = \frac{1}{2}(\hat{l}_+ + \hat{l}_-) \), we also have for the matrix representation

\[
\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) = \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (36)

Finally, from \( \hat{l}_y = \frac{\hat{l}_+ - \hat{l}_-}{2i} \) we get

\[
\hat{L}_y = \hbar \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}
\] (37)

1h. Compute the Wigner D-matrix elements

\[
d^{(l)}_{mk}(\beta) = \langle lm | e^{-\frac{i}{\hbar}\hat{L}_y} | lk \rangle
\]

for \( l = 1/2 \).

Answer: If we keep the order of the basis functions in the same order as in the previous question, we need to compute

\[
e^{-\frac{1}{\hbar}\beta \hat{L}_y} = e^{\frac{1}{\hbar}\beta A}
\] (38)

with

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] (39)

First, we calculate the eigenvalues of the matrix from

\[
\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0,
\] (40)

so \( \lambda_{\pm} = \pm i \). Note that the eigenvalues are imaginary, since the matrix \( A \) is anti-Hermitian. For \( \lambda_+ = i \), we find the eigenvector from

\[
(A - iI)e = 0
\] (41)
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\[
\begin{pmatrix}
-i & 1 \\
-1 & -i
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\] (42)

so

\[
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}.
\] (43)

For the eigenvalue \( \lambda_- = -i \) we find the eigenvector from

\[
\begin{pmatrix}
i & 1 \\
-1 & i
\end{pmatrix}
\begin{pmatrix}
d_1 \\
d_2
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\] (44)

so

\[
\begin{pmatrix}
d_1 \\
d_2
\end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}.
\] (45)

Note that the eigenvectors are orthogonal, since

\[
c_1^\dagger d_2 = (1 - i)
\begin{pmatrix} i \\ 1 \end{pmatrix} = i - i = 0.
\] (46)

We now construct a unitary matrix \( U \) from the normalized eigenvectors

\[
U = \frac{1}{\sqrt{2}} [c \ d] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ i \\ 1 \end{pmatrix}
\] (47)

so we can write the spectral decomposition of \( A \) as

\[
A = U \Lambda U^\dagger
\] (48)

where

\[
\Lambda = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\] (49)

We can now compute the \( l = 1/2 \) Wigner d-matrix from

\[
U e^{\pm i\beta A} U^\dagger = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
\begin{pmatrix} e^{\pm i\beta_i} & 0 \\ 0 & e^{\mp i\beta_i} \end{pmatrix}
\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
\] (50)

\[
= \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
\begin{pmatrix} e^{\pm i\beta_i} & -ie^{\pm i\beta_i} \\ -ie^{\mp i\beta_i} & e^{\mp i\beta_i} \end{pmatrix}
= \frac{1}{2} \begin{pmatrix} e^{\pm i\beta_i} + e^{-i\beta_i} & -ie^{\pm i\beta_i} + ie^{-i\beta_i} \\ ie^{\pm i\beta_i} - ie^{-i\beta_i} & e^{\pm i\beta_i} + e^{-i\beta_i} \end{pmatrix}
\] (51)

\[
= \begin{pmatrix} \cos \frac{\beta}{2} & \sin \frac{\beta}{2} \\ -\sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix}.
\] (52)

Note that since the matrix representations of the ladder operators are real, so the \( L_y \)-matrix is purely imaginary, the Wigner-d matrix is always real.

11. Show that the Wigner-D matrices satisfy the matrix representation property

\[
D^{(i)}(\hat{R}_1 \hat{R}_2) = D^{(i)}(\hat{R}_1) D^{(i)}(\hat{R}_2),
\] (53)

starting from the defining equation of the \( D \)-matrices.

Answer: The defining equation is

\[
\hat{R}_i |m\rangle = \sum_k |k\rangle D_{km}^{(i)}(\hat{R}_i)
\] (54)

so

\[
(\hat{R}_1 \hat{R}_2) |m\rangle = \sum_k |k\rangle D_{km}^{(i)}(\hat{R}_1 \hat{R}_2).
\] (55)
but also

\[ \hat{R}_1 \hat{R}_2 |lm \rangle = \hat{R}_1 \sum_{k'} |l'k' \rangle D^{(l)}_{k'm}(\hat{R}_2) \]
\[ = \sum_k \sum_{k'} |l'k' \rangle D^{(l)}_{kk'}(\hat{R}_1) D^{(l)}_{k'm}(\hat{R}_2) \]
\[ = \sum_k |l'k' \rangle \left[ D^{(l)}(\hat{R}_1) D^{(l)}(\hat{R}_2) \right]_{kk'} . \]

Comparing Eqs. (55) and (58) we find

\[ D^{(l)}_{kk'}(\hat{R}_1 \hat{R}_2) = \left[ D^{(l)}(\hat{R}_1) D^{(l)}(\hat{R}_2) \right]_{kk'} , \]

for \( k, m = -l, -l+1, \ldots, l \), which proves Eq. (53).