Mechanics 2, key equations

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October 11, 2026

1 Newton, Cartesian

Equations apply in 1D, 2D, and 3D,

$$\boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{1}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{d}{dt}\boldsymbol{r}(t) \tag{2}$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \frac{d^2}{dt^2}\boldsymbol{r}(t) \tag{3}$$

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + \boldsymbol{v}(0)t + \frac{1}{2}\boldsymbol{a}t^2. \tag{4}$$

Newton's 2nd law, N-D

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{5}$$

Linear momentum, Cartesian

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{6}$$

Newton's 2nd law for momentum

$$\dot{\boldsymbol{p}} = \boldsymbol{F}. \tag{7}$$

Conservative force

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r}). \tag{8}$$

Center-of-mass, n particles

$$\boldsymbol{r} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{r}_i}{\sum_{i=1}^{n} m_i}. \tag{9}$$

Total linear momentum

$$\boldsymbol{P} = \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} m_i \boldsymbol{v}_i. \tag{10}$$

No external forces

$$\boldsymbol{P} = M\dot{\boldsymbol{X}}, \quad (M = \textstyle\sum_i m_i). \tag{11}$$

2 Hamilton

Kinetic energy Cartesian, n particles

$$T = \sum_{i=1}^{n} m_i v_i^2 = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} \tag{12}$$

$$v_i^2 = \boldsymbol{v}_i \cdot \boldsymbol{v}_i \tag{13}$$

$$p_i^2 = \boldsymbol{p}_i \cdot \boldsymbol{p}_i. \tag{14}$$

Momentum, components $\alpha = x, y, z$,

$$p_{i,\alpha} \equiv \frac{\partial}{\partial v_{i,\alpha}} T(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n). \tag{15}$$

Potential, gravity

$$V = \sum_{i=1}^{n} m_i g z_i. \tag{16}$$

Harmonic potential, spring

$$V(r) = \frac{1}{2}k(r - r_e)^2. \tag{17}$$

Hamiltonian

$$H = T + V. \tag{18}$$

Generalized coordinates $(\boldsymbol{q}, \boldsymbol{p})$

$$p_i \equiv \frac{\partial}{\partial \dot{q}_i} T(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{19}$$

Hamilton's equation of motion

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(\boldsymbol{q}, \boldsymbol{p}) \tag{20}$$

$$\dot{p}_i = -\frac{\partial}{\partial q_i} H(\boldsymbol{q}, \boldsymbol{p}). \tag{21}$$

3 CoM coordinates

Two particles, relative coordinate

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1. \tag{22}$$

CoM to lab-frame

$$\boldsymbol{r}_1 = \boldsymbol{X} - \frac{m_2}{m_1 + m_2}\boldsymbol{r} \tag{23}$$

$$\boldsymbol{r}_2 = \boldsymbol{X} + \frac{m_1}{m_1 + m_2}\boldsymbol{r}. \tag{24}$$

Hamiltonian

$$H = \frac{P^2}{2M} + \frac{p_r^2}{2\mu} + V(\boldsymbol{r}), \tag{25}$$

with

$$M = m_1 + m_2 \tag{26}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{27}$$

Relative momentum

$$\boldsymbol{p}_r = \mu \boldsymbol{r} \tag{28}$$

$$p_r^2 = \boldsymbol{p}_r \cdot \boldsymbol{p}_r. \tag{29}$$

4 Polar coordinates, 2D

$$x = r\cos\phi \tag{30}$$

$$y = r\sin\phi. \tag{31}$$

Distance along circle, ϕ in rad,

$$s = r\phi. \tag{32}$$

Angular velocity

$$\omega = \dot{\phi}. \tag{33}$$

Moment of inertia, particle mass m,

$$I(r) = mr^2. \tag{34}$$

Kinetic energy

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}I(r)\omega^2. \tag{35}$$

Angular momentum

$$L = I\omega. \tag{36}$$

Radial momentum

$$p_r = m\dot{r}. \tag{37}$$

Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2I} + V(r,\phi). \tag{38}$$

Effective potential

$$V_L^{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r,\phi). \tag{39}$$

Hamilton's equation of motion

$$\dot{r} = \frac{p_r}{m} \tag{40}$$

$$\dot{p}_r = -\frac{\partial}{\partial r}V_L^{\text{eff}}(r) \tag{41}$$

$$\omega = \frac{L}{I(r)} \tag{42}$$

$$\dot{L} = -\frac{\partial}{\partial\phi}V(r,\phi). \tag{43}$$

5 Discs, cylinders

Cylinder and ring of mass M,

$$I(R) = MR^2. \tag{44}$$

Solid disc

$$I_{\text{disc}} = \frac{1}{2}MR^2 \tag{45}$$

No slip condition rolling cylinder

$$v_s = R\omega. \tag{46}$$

Kinetic energy

$$T = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_s^2 \tag{47}$$

Parallel axis theorem

$$\tilde{I} = I + MD^2. \tag{48}$$

6 Angular momentum, 3D

Angular momentum

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p} = \mu\boldsymbol{r}\times\boldsymbol{v}. \tag{49}$$

Effective potential, Hamiltonian, equations of motion, as for $2D$, with $m \to \mu$. Equation of motion $\boldsymbol{L}$, torque

$$\dot{\boldsymbol{L}} = \boldsymbol{r}\times\boldsymbol{F} = \boldsymbol{\tau}. \tag{50}$$

z-component

$$L_z = xp_y - yp_x = mrv_\perp = mr^2\omega. \tag{51}$$

Torque

$$\dot{L}_z = rF\sin\psi, \tag{52}$$

with ψ angle between $\boldsymbol{r}$ and $\boldsymbol{F}$.

Central force

$$\boldsymbol{F} = -\hat{\boldsymbol{r}}V'(r). \tag{53}$$

Gravitational potential

$$V_g(r) = -G\frac{m_1m_2}{r}. \tag{54}$$

Coulomb potential

$$V_{\text{Coulomb}}(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r}. \tag{55}$$

Minimum effective potential, $-c/r$

$$r_e = \frac{L^2}{\mu c}. \tag{56}$$

Orbital period, circular orbit

$$T_{\text{orb}} = \frac{2\pi}{\omega}. \tag{57}$$

Kepler's 3rd law

$$T_{\text{orb}}^2 = \frac{4\pi^2}{GM}a^3. \tag{58}$$

Kepler's 1st law, ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{59}$$

Distance between focal points is $2c$,

$$a^2 = b^2 + c^2. \tag{60}$$

Aphelion and perihelion

$$r_a = a + c \tag{61}$$

$$r_p = a - c \tag{62}$$

Eccentricity

$$e = c/a. \tag{63}$$

Kepler's 2nd law

$$\Delta A = \frac{1}{2}rv_\perp\Delta t = \frac{1}{2}\frac{L}{\mu}\Delta t. \tag{64}$$

7 Noether's theorem

Hamiltonian invariance and conservation law

- time $\leftrightarrow$ energy
- translation $\leftrightarrow$ momentum
- orientation $\leftrightarrow$ angular momentum

For translation of potential

$$V(\boldsymbol{r}+\delta\boldsymbol{r}) = V(\boldsymbol{r}) + \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}V(\boldsymbol{r}). \tag{65}$$

Rotation around $\hat{\boldsymbol{n}}$ angle $\delta\phi$

$$\delta\boldsymbol{r} = \delta\phi\,\hat{\boldsymbol{n}}\times\boldsymbol{r}. \tag{66}$$

8 Static equilibrium

General: minimum of potential.

$$\sum_i \boldsymbol{F}_i^{\text{ext}} = \boldsymbol{0} \tag{67}$$

$$\sum_i \boldsymbol{\tau}_i^{\text{ext}} = \boldsymbol{0}. \tag{68}$$

9 Fluid mechanics

Pressure

$$P = \frac{F}{A}. \tag{69}$$

Pressure as function of depth

$$P = P_0 + \rho h g. \tag{70}$$

Archimedes law, buoyant force

$$B = \rho V_{\text{sub}}\, g. \tag{71}$$

Potential energy due to buoyancy, center at depth D

$$V(D) = \frac{D^2}{2} A\rho g. \tag{72}$$

(Skip surface tension)

Flux

$$Q = vA. \tag{73}$$

Continuity equation, incompressible fluid

$$A_1 v_1 = A_2 v_2. \tag{74}$$

Bernoulli's equation

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2. \tag{75}$$

10 Poisson brackets (optional)

Definition

$$\{f,g\} \equiv \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right). \tag{76}$$

Time evolution property, $A(\boldsymbol{p},\boldsymbol{q})$

$$\dot{A} = \{A, H\}. \tag{77}$$

Hamilton's equations

$$\dot{q}_i = \{q_i, H\} \tag{78}$$

$$\dot{p}_i = \{p_i, H\}. \tag{79}$$

Translation

$$\frac{\partial A}{\partial q_i} = \{A, p_i\}. \tag{80}$$

Rotation, $(\boldsymbol{l} = \boldsymbol{r}\times\boldsymbol{p})$

$$A(\boldsymbol{r}+\epsilon\hat{\boldsymbol{n}}\times\boldsymbol{r}) = A(\boldsymbol{r}) + \epsilon\{A, \hat{\boldsymbol{n}}\cdot\boldsymbol{l}\}. \tag{81}$$

Poisson bracket rules, functions f, g, h. Antisymmetry

$$\{f,g\} = -\{g,f\}. \tag{82}$$

Linearity

$$\{f,\lambda g\} = \lambda\{f,g\} \tag{83}$$

$$\{f,g+h\} = \{f,g\} + \{f,h\}. \tag{84}$$

Leibniz rule

$$\{f,gh\} = g\{f,h\} + \{f,g\}h. \tag{85}$$

Noether's theorem: if H invariant under G,

$$\{H,G\} = 0 \tag{86}$$

then G conserved

$$\{G,H\} = 0 \tag{87}$$

and vice versa.