Mechanics

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Chapter 1

Linear motion

These lecture notes of the course "Mechanics, electricity, and Magnetism 2", as given in the first quarter of 2025. It does not replace the text book (Serway [1]), but it summarizes the equations and main ideas.

1.1 Classical mechanics in one dimension

In classical mechanics a system of n particles is defined by the **masses** of the particles, $\{m_i, i = 1, 2, \ldots, n\}$ and their **positions** as a function on time (t), $\{x_i(t), i = 1, 2, \ldots, n\}$. We will take $x_i(t)$ to be the Cartesian coordinate of particle i with respect to an **inertial** frame, as defined in **Newton's first law**. The function $x_i(t)$ is called the **trajectory** or also the **orbit**.

The time derivative of the position is the velocity

$$v(t) = \dot{x}(t) = \frac{d}{dt}x(t) \tag{1.1}$$

and the time derivative of the velocity is the acceleration

$$a = \dot{v} = \ddot{x}. \tag{1.2}$$

1.2 Newton's laws

Newton's first law defines an **inertial frame**. The main idea is that if there are no forces acting on a particle, it should be at rest or move with a constant velocity with respect to that frame.

Newton's second law relates the acceleration to the forces acting on the particle

$$F = ma. \tag{1.3}$$

If several forces F_i are acting on a particle than the acceleration is is related to the sum of forces

$$\sum F = ma. \tag{1.4}$$

Newton's third law concerns the interaction of two particles. If particle 1 exerts of force F on particle 2, then particle 1 experiences a force $-F$ as a result of the interaction.

1.3 Calculating a trajectory

With Newton's laws we can learn about forces by observing trajectories. A particle with initial velocity $v(t_i) = v_i$ experiencing a constant force F. From Newton's second law we get the time derivative of the velocity

$$\dot{v}(t) \equiv a(t) = \frac{1}{m}F(t), \tag{1.5}$$

and by integration we find

$$v(t) = v_i + \frac{1}{m}\int_{t_i}^{t} F(t)dt. \tag{1.6}$$

Once we know the velocity at every time t, we can compute the trajectory if we know the position $x(t_i) = r_i$ by integrating again

$$x(t_f) = x_i + \int_{t_i}^{t_f} v(t)dt. \tag{1.7}$$

When the force is actually constant, $F(t) = F_g$, we find that the velocity changes linearly in time

$$v(t) = v_i + \frac{t - t_i}{m} F_g. \tag{1.8}$$

An example would be the trajectory of particle dropping under the force of **gravity**. Using z rather than x for vertical motion we find that if initially $z(t_i) = z_i$ the trajectory is

$$z(t) = z_i + v_i(t_f - t_i) + \frac{(t - t_i)^2}{2m} F_g. \tag{1.9}$$

From experiment we find that the trajectory is independent of the mass and the force is pointing down, so we conclude that

$$F_g = -mg, \tag{1.10}$$

where $g \approx 9.8\ \mathrm{m/s}^2$, so that the acceleration does not depend on the mass, and matches observation

$$a = \ddot{z} = -g. \tag{1.11}$$

If the force is constant or if it depends on time we can find the trajectory by integration. It is also possible that the force depends on the position, $F_h = F_h(z)$. For a mass attached to a spring we can have, e.g.,

$$F_h(z) = -k(z - z_e), \tag{1.12}$$

where k is the **force constant**, and z_e is the **equilibrium position** where the force exerted by the spring is zero. By Newton's second law we know that the **sum of forces** determines the trajectory, so if we already know the force of gravity, we can determine the force constant k by finding the value of $z = z_s$ for which the mass is stationary

$$F(z_s) + F_g = 0 \tag{1.13}$$

$$-k(z_s - z_e) - mg = 0 \tag{1.14}$$

so

$$k = \frac{mg}{z_e - z_s}. \tag{1.15}$$

Note that by Newton's second law, the trajectory depends on the **sum of all forces** acting on the particle. This is very powerful: every time a trajectory does not match what we expect based on the forces we already know, we learn about a new force.

Now that we know the force as a function of position, we can try to predict the trajectory if we pull the mass to an initial position z_i and give it an initial velocity v_i. Define the sum of forces

$$F(z) = F_h(z) + F_g \tag{1.16}$$

we need to solve

$$F(z) = m\ddot{z}. \tag{1.17}$$

Integration over time as we did before no longer works, since we would need to $z(t)$ to find $F(t) = F(z(t))$. Instead, we now have a second order differential equation with two initial conditions. In our example of a spring there is a simple analytic solution, but in general numeric methods are needed. Sometimes it helps to turn the second order differential equation into two **coupled** first order differential equations

$$\dot{z}(t) = v(t) \tag{1.18}$$

$$\dot{v}(t) = \frac{1}{m} F(z). \tag{1.19}$$

In this lecture our aim is not to solve differential equations, but rather to try to simplify the problem as much as possible by using **conservation laws**. Usually, this will not give us the exact trajectory, but we may answer a simpler question, like what is the highest and lowest point of the mass on the spring, or later, what is the velocity of particles after a collision if we know their velocities before the collision.

1.4 Energy conservation

We can derive conservation laws by integrating Newton's second law. First, let us assume that we have a force $F(z)$ that depends on **position** only. Integrating Newton's second law then gives

$$\int_{z_i}^{z_f} F(z)dz = \int_{z_i}^{z_f} ma\, dz. \tag{1.20}$$

To integrate the left-hand-side (lhs) we define $-V(z)$ as the primitive of the force, i.e.,

$$F(z) = -\frac{d}{dz}V(z). \tag{1.21}$$

With these assumptions the integral is

$$\int_{z_i}^{z_f} F(z)dz = -\int_{z_i}^{z_f} \frac{d}{dz}V(z)\, dz = V(z_i) - V(z_f). \tag{1.22}$$

Not every force be we written as the derivative of a function that only depends on the position, but if it does, we call the force **conservative**. The function $V(z)$ is called **potential energy**. In the example of gravity we find potential energy

$$V_g(z) = mgz, \tag{1.23}$$

which says that the potential energy is larger the higher you go, and the bigger the mass, which may sound reasonable. The point of using the word **energy** though is that the total energy is **conserved** and that you may **convert** one form of energy into another.

Next, we solve the right-hand-side (rsh) of Eq. (1.20) Here comes a little **trick**: we can change an integral over **position** into an integral **time** using

$$\frac{dz}{dt} = v(t) \tag{1.24}$$

so

$$dz = vdt \tag{1.25}$$

At the same time, we have to change the limits of the integral from initial and final **position** to initial and final **time**, so

$$\int_{z_i}^{z_f} ma\, dz = m\int_{t_i}^{t_f} \dot{v}vdt \tag{1.26}$$

Next, we use

$$\frac{d}{dt}v^2 = 2v\dot{v}, \tag{1.27}$$

so the rhs of Eq. (1.26) becomes

$$\frac{1}{2}m\int_{t_i}^{t_f}\left(\frac{d}{dt}v^2\right)dt = \frac{1}{2}m(v_f^2 - v_i^2). \tag{1.28}$$

Substituting this result into the rhs of Eq. (1.20), expressing the lhs in terms of potential energy and rearranging the equation such that everything that depends on the initial state is on the left and everything that depends on the final state is on the right gives

$$\frac{1}{2}mv_i^2 + V(z_i) = \frac{1}{2}mv_f^2 + V(z_f). \tag{1.29}$$

This equation motivates the definition of **kinetic energy**

$$T = \frac{1}{2}mv^2, \tag{1.30}$$

so that for **conservative forces** we have a total energy

$$E = T + V(z), \tag{1.31}$$

which is conserved.

The integral over the force in Eq. (1.20) is called the **work** done on the particle. The definition of work is more general though, since it also applies to non-conservative forces, which we will discuss later.

1.5 Linear momentum

Integrating Newton's second law over position gave us the energy conservation law for converting between potential and kinetic energy. Now, let's see what we get from integrating Newton's second law over time

$$\int_{t_i}^{t_f} F(t)dt = \int_{t_i}^{t_f} ma\,dt. \tag{1.32}$$

First, let's do the rhs. Using $a = \dot{v}$ gives

$$\int_{t_i}^{t_f} ma\,dt = m\int_{t_i}^{t_f} \dot{v}dt = m(v_f - v_i). \tag{1.33}$$

We do not yet have a conservation law, but this equation motivates the definition of **linear momentum**

$$p \equiv mv, \tag{1.34}$$

so the rhs is the **change in linear momentum** between the initial and final time. It may seem there is not much we can do on the lhs of Eq. (1.32), so we can give it a name: **impulse**,

$$I \equiv \int_{t_i}^{t_f} F(t)dt = p_f - p_i. \tag{1.35}$$

The good thing about this quantity is that it is defined for an arbitrary force $F(t)$, i.e., it is not restricted to conservative forces. As long as the particle has a trajectory, $z(t)$, it has some acceleration at every time t, and we find the force from Newton's second law. Thus, we may also, write

$$I \equiv \int_{t_i}^{t_f} F\,dt, \tag{1.36}$$

where now the force may be know directly as $F(t)$, or implicitly as $F[z(t)]$. This is the definition given in Serway.

We also see that if the force is zero the impulse is zero, and linear momentum is conserved. However, we already knew from Newton's second law that if there is no force the velocity is constant, and so the linear momentum is constant. Still, we now find that if the force itself is not zero, but the integral over time, i.e. the impulse is zero, the linear momentum is conserved. Furthermore, we can define an **average force**

$$\overline{F} = \frac{\int_{t_i}^{t_f} F(t)dt}{t_f - t_i}. \tag{1.37}$$

Then by giving the average force $\overline{F}$ and the time interval $t_f - t_i$ we have specified the impulse

$$I = (t_f - t_i)\overline{F}. \tag{1.38}$$

Note that here by average, we mean **average over time**.

Many problems are easier when using momenta, rather than velocities, so we already rewrite Newton's second equation in momenta.

$$\sum F = \dot{p}. \tag{1.39}$$

The concepts of impulse and momentum turn into a real conservation law if we consider the collisions of **two particles**

1.6 Conservation of linear momentum

Let's consider two particles, labeled 1 and 2, which move along the same straight line with trajectories $x_1(t)$ and $x_2(t)$ and which have masses m_1 and m_2, respectively. From Newton's third law, we know that

$$F_{12} = -F_{21}, \tag{1.40}$$

where F_{12} is the force exerted by particle 1 on particle 2, and F_{21} is the force exerted by particle 2 on particle 1. This immediately tells us that the impulses on the particles due to their interaction are opposite in sign,

$$p_{1,f} - p_{1,i} = -(p_{2,f} - p_{2,i}). \tag{1.41}$$

We can rewrite this as the law of **conservation of linear momentum**

$$P \equiv p_{1,i} + p_{2,i} = p_{1,f} + p_{2,f}, \tag{1.42}$$

so the total linear momentum before the collision, P, is equal to the total linear momentum after the collision.

We can also express this relation using velocities

$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}. \tag{1.43}$$

1.7 Elastic collisions in 1D

Assuming the velocities of **two particles** before the collision are known, we try to find the velocities of the two particles after the collision. Having two unknowns, we can solve the problem if we have two independent equations. We can certainly use conservation of linear momentum [Eq. (1.42)], since it was derived from the impulse [Eq. (1.36)], where we did not make any assumption about the origin of the force. Instead of coordinates $z_1(t)$ and $z_2(t)$ we will use $x_1(t)$ and $x_2(t)$, because we are not thinking about gravity here, but of course this choice is arbitrary.

To continue, we assume that the forces between the particles are **conservative**. This means that we can compute the forces from a potential. For two particles a potential energy is function of both coordinates, so we have $V_{12}(x_1, x_2)$, and the forces are given by

$$F_1(x_1, x_2) = -\frac{\partial}{\partial x_1} V_{12}(x_1, x_2) \tag{1.44}$$

$$F_2(x_1, x_2) = -\frac{\partial}{\partial x_2} V_{12}(x_1, x_2) \tag{1.45}$$

If the force is the result of, e.g., a spring, the potential will only depend on the distance between the particles, $x = x_2 - x_1$,

$$V_{12}(x_1, x_2) = V(x_2 - x_1) \tag{1.46}$$

Then we find

$$F_1(x_1, x_2) = -\frac{\partial}{\partial x_1} V(x) = -\frac{dx}{dx_1}\frac{d}{dx} V(x) = V'(x) \tag{1.47}$$

and

$$F_2(x_1, x_2) = -\frac{\partial}{\partial x_2} V(x) = -V'(x) = -F_1(x_1, x_2). \tag{1.48}$$

Note that we just recovered Newton's third law for the special case of conservative forces! We now assume that the potential is zero when the particles are sufficiently far apart. That means we can take time t_i sufficiently long before the collision, so the energy of the system is just the sum of the kinetic energies of the particles. We take time t_f when the particles are moving apart and the potential energy $V(x_2 - x_1)$ has again dropped to zero, so we have conservation of kinetic energy

$$\frac{1}{2}m_1v_{1,i}^2+\frac{1}{2}m_2v_{2,i}^2 = \frac{1}{2}m_1v_{1,f}^2+\frac{1}{2}m_2v_{2,f}^2. \quad (1.49)$$

Strictly, we only derived above that the sum of kinetic and potential energy for a **single particle** is conserved. We will come back to this later and solve the **elastic collision** problem by solving the two equations with the two unknowns.

Since conservation of linear momentum is the simplest when written in terms of linear momenta, we rewrite the conservation of kinetic energy (for elastic collisions) also using momenta. In the final step of the calculation we will convert the momenta back to velocities. For the kinetic energy of a single particle we have

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m} \quad (1.50)$$

so for two particles we have

$$E = T_1 + T_2 = \frac{p_{1,i}^2}{2m_1} + \frac{p_{2,i}^2}{2m_2} = \frac{p_{1,f}^2}{2m_1} + \frac{p_{2,f}^2}{2m_2}. \quad (1.51)$$

We can turn this equation with two unknowns, $p_{1,f}$ and $p_{2,f}$, into a single equation with one unknown by substituting $p_{2,f} = P - p_{1,f}$ when we find the total linear momenta from $P = p_{1,i} + p_{2,i}$. Thus, we need to solve an equation quadratic in $p_{1,f}$,

$$E = \frac{p_{1,f}^2}{2m_1} + \frac{(P - p_{1,f})^2}{2m_2}. \quad (1.52)$$

To work out the general expression is quite a bit of work, although when you plug in the numbers this approach may be ok. The book (Serway) solves the problem using velocities rather than momenta in the derivation, but it shows a little trick make deriving the general expression easier, that we can use too. We rewrite the energy conservation law as

$$\frac{p_{1,f}^2 - p_{1,i}^2}{2m_1} = -\frac{p_{2,f}^2 - p_{2,i}^2}{2m_2} \quad (1.53)$$

so we have the change in kinetic energy of the particle on the left, and minus the change in kinetic energy of particle 2 on the right. Next, we factorize the difference of squares

$$\frac{(p_{1,f} + p_{1,i})(p_{1,f} - p_{1,i})}{2m_1} = -\frac{(p_{2,f} + p_{2,i})(p_{2,f} - p_{2,i})}{2m_2}. \quad (1.54)$$

We can now use the conservation of linear momentum [Eq. (1.41)] to divide the lhs by the change in linear momentum of particle 1 and the rhs by minus the change in the linear momentum of particle two. This way we get two linear equations with two unknowns, which are just a bit easier to solve than a single linear equation with one unknown

$$p_{1,f} - p_{1,i} = -(p_{2,f} - p_{2,i}) \quad (1.55)$$

$$\frac{p_{1,f} + p_{1,i}}{2m_1} = \frac{p_{2,f} + p_{2,i}}{2m_2}. \quad (1.56)$$

Multiplying the second equation with $2m_2$ and adding the equations gives a linear equation in $p_{1,f}$

$$(\frac{m_2}{m_1} + 1)p_{1,f} + (\frac{m_2}{m_1} - 1)p_{1,i} = 2p_{2,i} \quad (1.57)$$

Multiplying with m_1 and rearranging gives

$$p_{1,f} = \frac{(m_1 - m_2)p_{1,i} + 2m_1p_{2,i}}{M}, \quad (1.58)$$

where $M = m_1 + m_2$ is the **total mass**. Dividing by m_1 we can rewrite this in velocities

$$v_{1,f} = \frac{(m_1 - m_2)v_{1,i} + 2m_2v_{2,i}}{M}. \quad (1.59)$$

By multiplying Eq. (1.56) with m_1 and adding the two equations, we can derive in a similar way the expression for $p_{2,f}$. Alternatively, we can use conservation of linear momentum

$$p_{2,f} = \underbrace{p_{1,i} + p_{2,i}}_{\text{total momentum}} - p_{1,f} \tag{1.60}$$

so

$$\begin{aligned} p_{2,f} &= \frac{(m_1+m_2)(p_{1,i}+p_{2,i}) + (m_2-m_1)p_{1,i} - 2m_1p_{2,i}}{M} \\ &= \frac{2m_2p_{1,i} + (m_2-m_1)p_{2,i}}{M}. \end{aligned} \tag{1.61}$$

Dividing by m_1 and rewriting in velocities gives

$$v_{2,f} = \frac{2m_1v_{1,i} + (m_2-m_1)v_{2,i}}{M}. \tag{1.62}$$

For the special case of $m_1 = m_2$ this simplifies to

$$p_{1,f} = p_{2,i} \tag{1.63}$$

$$p_{2,f} = p_{1,i} \tag{1.64}$$

so the momenta swap. Since the masses are equal we also get $v_{1,f} = v_{2,i}$ and $v_{2,f} = v_{1,i}$.

If initially particle 2 is at rest, $p_{2,i} = 0$, we find

$$p_{1,f} = \frac{m_1 - m_2}{M} p_{1,i} \tag{1.65}$$

$$p_{2,f} = \frac{2m_2}{M} p_{1,i}. \tag{1.66}$$

and also

$$v_{1,f} = \frac{m_1 - m_2}{M} v_{1,i} \tag{1.67}$$

$$v_{2,f} = \frac{2m_2}{M} v_{1,i}. \tag{1.68}$$

If, in addition, $m_1 \gg m_2$,

$$v_{1,f} = v_{1,i} \tag{1.69}$$

$$v_{2,f} = 2v_{1,i}. \tag{1.70}$$

1.8 Energy conservation for non-conservative forces

We may repeat the derivation of section 1.4, but for the lhs of Eq. (1.20), the **work**

$$W = \int_{z_i}^{z_f} F dz \tag{1.71}$$

we do not assume the force is related to a potential energy. The rhs of Eq. (1.20) still gives the same, so we find that the work gives the difference in kinetic energy

$$W = T_f - T_i. \tag{1.72}$$

An example of a non-conservative force is **friction**. A particle (or object) sliding over a surface until it comes to rest has lost all its kinetic energy. We could have made a model of the surface consisting of particles (atoms or molecules), interacting through a potential function that depends on all the coordinates. Then, if would take into account the kinetic energy of the atoms in the surface, mechanical energy would be conserved. If we only want to include the sliding particle in the model, we call all the energy that has gone into the surface **heat** and **total energy** defined as **kinetic energy** of the sliding particle **plus** the heat is again conserved. Thus, the term **heat** appears when we want to sweep a lot of detail under the rug. This approach is elaborated in **thermodynamics**, where the concepts of **work** and **heat** are central, but the microscopic detail of trajectories of particles is left out. In **statistical mechanics** a connection is made between **thermodynamics** and a microscopic description of the system. For now, we say that the work done by non-conservative forces, e.g., friction, on the (sliding) particle, is converted to heat.

1.9 Inelastic collisions

In an **inelastic collision**, part of the kinetic energy is lost. In a **perfectly inelastic collision**, the two particles stick together.

When an atom collides with a molecule it may happen that the molecules get rotationally or vibrationally excited. By the laws of quantum mechanics this will happen in discrete energy **quanta**. This energy is no longer available for the translational kinetic energy of the atom and molecule after the collision, so we would call such a collisions **inelastic**. Since the energy in the molecule after the collision may not be distributed among all possible vibrations and rotations according to thermodynamics we would not call this energy **heat**. However, if the molecule is large the energy may redistribute eventually, and we may say that the collision has heated the molecule.

In a **perfectly inelastic** collision, where the molecules stick together, we have only one unknown the velocity of the particles after the collision. Since the **law of conservation of linear momentum** applies to both conservative and nonconservative forces we can use it to solve this problem:

$$p_{1,i} + p_{2,i} = p_f \tag{1.73}$$

where

$$p_f = (m_1 + m_2)v_f \tag{1.74}$$

The solution is

$$v_f = \frac{m_1 v_{1,i} + m_2 v_{2,i}}{m_1 + m_2}. \tag{1.75}$$

1.10 Center of mass

The center of mass (C.O.M.) is the mass weighted average of the position of the particles

$$X(t) = \frac{m_1 x_1(t) + m_2 x_2(t)}{M}, \tag{1.76}$$

where again $M = m_1 + m_2$. The velocity of the C.O.M. is the time derivative

$$V(t) = \dot{X}(t) = \frac{m_1 v_1(t) + m_2 v_2(t)}{M} = \frac{p_1 + p_2}{M} = \frac{P}{M}. \tag{1.77}$$

So, for the total momentum we have

$$P = MV. \tag{1.78}$$

Also, the equation of motion of the C.O.M. is the same as for a particle of mass M, and we find

$$\dot{P} = \dot{p}_1 + \dot{p}_2 = \sum F_1 + \sum F_2 = \sum F, \tag{1.79}$$

where the sum over forces includes all forces acting on either particle 1, or particle 2. The forces that arise from the interactions between the particles cancel by Newton's third law, so in fact, we may sum over external forces only

$$\dot{P} = \sum F_{\text{ext}}, \tag{1.80}$$

and if there are no external forces, as in the collision problems we are studying, we find the P is constant, as before.

Note that the C.O.M. velocity V is equal to $v_{1,f} = v_{2,f}$ in a **perfectly inelastic** collision.

1.11 Center of mass frame

The equation of motion of the C.O.M. is relatively simple, since it does not depend on internal forces. Therefore, it is worthwhile to switch for coordinates x_1 and x_2 to a new set of coordinates, the C.O.M. coordinates

$$X = \frac{m_1 x_1 + m_2 x_2}{M} \tag{1.81}$$

and a relative coordinate

$$x = x_2 - x_1. \tag{1.82}$$

We can solve these equations for x_1 and x_2,

$$x_1 = X - \frac{m_2 x}{M} \tag{1.83}$$

$$x_2 = X + \frac{m_1 x}{M}. \tag{1.84}$$

This result is easily verified by substituting it back into Eqs. (1.81) and (1.82).

We already have an equation of motion for the C.O.M. X. For the distance between the particles we have

$$\ddot{x} = \ddot{x}_2 - \ddot{x}_1 \tag{1.85}$$

$$= \frac{F_2}{m_2} - \frac{F_1}{m_1}. \tag{1.86}$$

Now consider **conservative forces** so we have

$$F_1 = -F_2 = V'(x) \equiv F, \tag{1.87}$$

where we define the internal force F. Then

$$\ddot{x} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) F = \frac{m_1 + m_2}{m_1 m_2} F. \tag{1.88}$$

Thus, by defining the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.89}$$

(does μ have the dimension mass?), we find Newton's equation of motion for $x(t)$,

$$F = \mu \ddot{x}. \tag{1.90}$$

Thus, equations of motion for $X(t)$ and $x(t)$ are not coupled, with $X(t)$ depending on external forces, and $x(t)$ depending on internal forces. The problem of solving $x(t)$ is identical to solving the motion of a particle with mass μ driven by potential energy $V(x)$. Hence, we can define a velocity $v = \dot{x}$ and momentum p is

$$p = \mu v. \tag{1.91}$$

After solving the internal motion $x(t)$, $v(t)$, and $p(t)$ we can find the momenta of the two particles

$$\begin{aligned} p_1 &= m_1 \dot{x}_1 = m_1 \dot{X} - \frac{m_1 m_2}{M}\frac{p}{\mu} = \frac{m_1}{M} P - p \\ p_2 &= m_2 \dot{x}_2 = m_2 \dot{X} + \frac{m_2 m_1}{M}\frac{p}{\mu} = \frac{m_2}{M} P + p \end{aligned} \tag{1.92}$$

and for the velocities

$$v_1 = \frac{p_1}{m_1} = V - \frac{p}{m_1} \tag{1.93}$$

$$v_2 = \frac{p_2}{m_2} = V + \frac{p}{m_2}. \tag{1.94}$$

The kinetic energy, expressed in C.O.M. momenta is

$$T = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{p^2}{2\mu}. \tag{1.95}$$

So the kinetic energy is the sum of a contribution from the C.O.M. and a contribution from the relative motion.

1.12 Elastic collision in 1D, using C.O.M. frame

We assume that the initial velocities, $v_{1,i}$ and $v_{2,i}$ are known. For the C.O.M. velocity we find with Eq. (1.77)

$$V = \frac{m_1}{M} v_{1,i} + \frac{m_2}{M} v_{2,i}. \tag{1.96}$$

The initial relative velocity is

$$v_i = v_{2,i} - v_{1,i}. \tag{1.97}$$

The initial relative momentum is

$$p_i = \mu v_i = \frac{m_1 m_2}{m_1 + m_2} v_i. \tag{1.98}$$

Due to the collision, **the relative velocity and momentum change sign**:

$$p_f = -p_i. \tag{1.99}$$

With Eqs. (1.93) and (1.94) we find the final velocities in the lab frame

$$v_{1,f} = V - \frac{p_f}{m_1} \tag{1.100}$$

$$v_{2,f} = V + \frac{p_f}{m_2}. \tag{1.101}$$

Exercise: check that these results match Eqs. (1.59) and (1.62)

Note: in quantum mechanics the equations of motion are very different. Still, the C.O.M. motion and the relative motion can be solved separately.

1.13 Motion in 2D

Much of what we did so far can readily be extended to 2D. We must keep in mind though, that a trajectory now has two components

$$\boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{1.102}$$

and the velocity $\boldsymbol{v}(t)$, the acceleration $\boldsymbol{a}(t)$, linear momentum $\boldsymbol{p}(t)$, the force $\boldsymbol{F}(t)$, and the impulse $\boldsymbol{I}$, now all become vectors with an x and a y component. Masses and energies remain scalars. So Newton's second law in 2D is

$$\boldsymbol{F} = m\boldsymbol{a} = \dot{\boldsymbol{p}}. \tag{1.103}$$

For a **conservative force** the derivative of the potential becomes a gradient

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r}) = -\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} V(\boldsymbol{r}). \tag{1.104}$$

Integrating Newton's second law over time now requires integrals over x and y, and the results are similar, e.g., conservation of linear momentum turns into conservation of linear momentum in the x- and y- direction.

To derive the relation between work and kinetic energy we need

$$\int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} m\boldsymbol{a} \cdot d\boldsymbol{r} \tag{1.105}$$

and to turn the integral over position into an integral over we need use

$$d\boldsymbol{r} = \boldsymbol{v}\, dt \tag{1.106}$$

and

$$\boldsymbol{a} \cdot \boldsymbol{v} = \frac{1}{2}\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v}). \tag{1.107}$$

For the integral over the force we use

$$\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{v}\, dt \tag{1.108}$$

This can be rewritten as a **total derivative** with respect to t,

$$\frac{d}{dt}V(\boldsymbol{r}) = \left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}\right) V(\boldsymbol{r}) \tag{1.109}$$

$$= \boldsymbol{v} \cdot \boldsymbol{\nabla} V(\boldsymbol{r}) = -\boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{v} \tag{1.110}$$

so

$$\int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{t_i}^{t_f} \frac{d}{dt}V(\boldsymbol{r})dt = V(\boldsymbol{r}_i) - V(\boldsymbol{r}_f). \tag{1.111}$$

The same method can be used to prove the **conservation of mechanical energy for two particles** moving on a straight line interacting with a **conservative force**, as in Eq. (1.49).

1.14 C.O.M. frame in 2D

The C.O.M. and relative coordinates are

$$\boldsymbol{X} = \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2 \tag{1.112}$$

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1. \tag{1.113}$$

Transforming back to the lab frame

$$\boldsymbol{r}_1 = \boldsymbol{X} - \frac{m_2}{M}\boldsymbol{r} \tag{1.114}$$

$$\boldsymbol{r}_2 = \boldsymbol{X} + \frac{m_1}{M}\boldsymbol{r}. \tag{1.115}$$

For the relative momentum we have

$$\boldsymbol{p} = \mu\boldsymbol{v} = \mu\dot{\boldsymbol{r}} \tag{1.116}$$

and the momentum associated with the C.O.M. motion is the total linear momentum

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = M\boldsymbol{V}. \tag{1.117}$$

After solving $\boldsymbol{r}(t)$, $\boldsymbol{v}(t)$, and $\boldsymbol{p}(t)$ we can transform the result back to the lab frame, as in Eqs. (1.92)

$$\boldsymbol{p}_1 = m_1\dot{\boldsymbol{r}}_1 = \frac{m_1}{M}\boldsymbol{P} - \boldsymbol{p} \tag{1.118}$$

$$\boldsymbol{p}_2 = m_2\dot{\boldsymbol{r}}_2 = \frac{m_2}{M}\boldsymbol{P} + \boldsymbol{p} \tag{1.119}$$

and for the velocities

$$\boldsymbol{v}_1 = \frac{\boldsymbol{p}_1}{m_1} = \boldsymbol{V} - \frac{\boldsymbol{p}}{m_1} \tag{1.120}$$

$$\boldsymbol{v}_2 = \frac{\boldsymbol{p}_2}{m_2} = \boldsymbol{V} + \frac{\boldsymbol{p}}{m_2}. \tag{1.121}$$

The total kinetic energy is, as before

$$T = \frac{P^2}{2M} + \frac{p^2}{2\mu}, \tag{1.122}$$

except that now, $P^2 = \boldsymbol{P}\cdot\boldsymbol{P}$ and $p^2 = \boldsymbol{p}\cdot\boldsymbol{p}$.

1.15 Perfectly inelastic, 2D, C.O.M. frame

After the collision the stick to each other, so

$$\boldsymbol{v}_{1,f} = \boldsymbol{v}_{2,f} = \boldsymbol{V} = \frac{m_1\boldsymbol{v}_{1,i} + m_2\boldsymbol{v}_{2,i}}{m_1 + m_2}. \tag{1.123}$$

1.16 Elastic, 2D. C.O.M. frame

We have four unknowns, the x and y components of $\boldsymbol{v}_{1,f}$ and $\boldsymbol{v}_{2,f}$. However, we only have three conservation laws: linear momentum in the x-direction, linear momentum in the y-direction, and mechanical energy conservation.

The C.O.M. linear momentum $\boldsymbol{P}$ takes care of the linear momentum conservation. Conservation of kinetic energy [Eq. (1.122)] gives

$$\frac{p_i^2}{2\mu} = \frac{p_f^2}{2\mu}, \tag{1.124}$$

since the C.O.M. contributions cancel, so this gives as the length of the vector $|\boldsymbol{p}_f| = p_f$. If, in an experiment we measure the directions after the collision, we can solve for the momenta and velocities of both particles.

$$\boldsymbol{p}_f = p_f\hat{\boldsymbol{e}}_f, \tag{1.125}$$

where $\hat{\boldsymbol{e}}_f$ is a unit vector. We may specify it with angle ϕ_f as

$$\boldsymbol{e}_f = \begin{pmatrix}\cos\phi_f \\ \sin\phi_f\end{pmatrix}. \tag{1.126}$$

With the equations in section 1.14 we can than find the velocities $\boldsymbol{v}_{1,f}$ and $\boldsymbol{v}_{2,f}$.

Chapter 2

Circular motion

We consider a particle moving on a circle with radius r. Its position in 2D as a function of time t is given by

$$\boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}, \tag{2.1}$$

where (x, y) are the Cartesian coordinates of the particle, and (r, θ) the polar coordinates. Unless otherwise specified, we assume angles are given in **radians**. We take the center of the circle as the origin of the coordinate system. The radius r is assumed to be constant, so the time dependent position follows from the time-dependent polar angle $\theta(t)$. The **circumference** of the circle is $2\pi r$. The **distance traveled** along the circle at time t is

$$s(t) = r\theta(t) \tag{2.2}$$

if we start at $\theta(0) = 0$. We define the unit vector

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{2.3}$$

where the hat on $\boldsymbol{r}$ indicates that the vector has length one $|\hat{\boldsymbol{r}}| = 1$, since

$$|\hat{\boldsymbol{r}}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1. \tag{2.4}$$

So we may also write

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}. \tag{2.5}$$

Thus, for circular motion θ takes the role of the position z in linear motion. Note, however, that this assumes the origin and the radius of the circle are known.

The velocity of the particle is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{d\theta}{dt}\frac{d}{d\theta}\boldsymbol{r} = r\dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{2.6}$$

The particle is always moving in a direction perpendicular to $\boldsymbol{r}$. We define the unit vector

$$\hat{\boldsymbol{r}}_\perp \equiv \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{2.7}$$

We define the angular velocity

$$\omega \equiv \dot{\theta}, \tag{2.8}$$

so the velocity is

$$\boldsymbol{v} = r\omega\hat{\boldsymbol{r}}_\perp, \tag{2.9}$$

and the **speed** of the particle

$$v = |\boldsymbol{v}| = r\omega. \tag{2.10}$$

Note that this is also the time derivative of the distance traveled [Eq. (2.2)]

$$\dot{s} = r\dot{\theta} = r\omega = v. \tag{2.11}$$

The **acceleration** is

$$\boldsymbol{a} = \ddot{r} = \dot{v} = r\dot{\omega}\hat{\boldsymbol{r}}_\perp + r\omega\frac{d}{dt}\hat{\boldsymbol{r}}_\perp \tag{2.12}$$

For the time derivative of perpendicular direction we find

$$\frac{d}{dt}\hat{\boldsymbol{r}}_\perp = \frac{d\theta}{dt}\frac{d}{d\theta}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} = -\omega\hat{\boldsymbol{r}}. \tag{2.13}$$

With the **angular acceleration** define by

$$\alpha \equiv \dot{\omega}, \tag{2.14}$$

we find for the total linear acceleration

$$\boldsymbol{a} = r\alpha\hat{\boldsymbol{r}}_\perp - r\omega^2\hat{\boldsymbol{r}}. \tag{2.15}$$

Thus, there are two contributions, which are orthogonal to each other. The first term is called the **tangential acceleration**, $\boldsymbol{a}_t$, and its magnitude is

$$a_t = |\boldsymbol{a}_t| = r\alpha. \tag{2.16}$$

The second term is the **radial acceleration**, $\boldsymbol{a}_r$, which is pointing inward along $\hat{\boldsymbol{r}}$, and its magnitude is the **centripetal acceleration**

$$a_c \equiv a_r = |\boldsymbol{a}_r| = r\omega^2 = \frac{v^2}{r}, \tag{2.17}$$

where the last step follows from $v = r\omega$ [Eq. (2.10)]. Summarizing, the **total linear acceleration** is

$$\boldsymbol{a} = a_t\hat{\boldsymbol{r}}_\perp - a_c\hat{\boldsymbol{r}}. \tag{2.18}$$

2.1 Equation of motion

If the angular speed ω is constant then the tangential acceleration a_t is zero. However, if ω is not zero, there will still be a centripetal acceleration. By Newton's second law there must be a corresponding force, and this would be the force that keeps the particle in the circular orbit, e.g., a string pulling the mass towards the center.

Thus, we can compute the tangential acceleration by taking into account only the tangential part of the force that accelerates the particle along the circle, $\boldsymbol{F}_t$. Note that any radial component, $\boldsymbol{F}_r$, of an additional force will simply be countered by whatever force along $\hat{\boldsymbol{r}}$ is keeping the particle on the circle. Thus, in general we would decompose a force as

$$\boldsymbol{F} = \boldsymbol{F}_t + \boldsymbol{F}_r = \hat{\boldsymbol{r}}_\perp(\hat{\boldsymbol{r}}_\perp \cdot \boldsymbol{F}) + \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{F}). \tag{2.19}$$

From Newton's second law we have

$$\boldsymbol{F}_t = m\boldsymbol{a}_t, \tag{2.20}$$

or, for the magnitudes

$$F_t = ma_t = mr\alpha. \tag{2.21}$$

2.1.1 Kinetic energy

The kinetic energy of a particle with mass m, angular velocity ω, in a circular orbit with radius r, is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2, \tag{2.22}$$

where in the last step we introduced the **moment of inertia**,

$$I = mr^2. \tag{2.23}$$

Thus, with ω the angular analogy of linear velocity v, the moment of inertia would be the angular analogy of mass.

Table 2.1: Linear vs angular motion

linear		angular	
Cartesian coordinate	z	θ	polar angle
Velocity	$v = r\omega$	ω	angular velocity
Acceleration	$a = r\alpha$	α	angular acceleration
Mass	m	$I = mr^2$	moment of inertia
Force	F	$\tau = rF_t$	torque
Newton's 2nd	$F = ma$	$\tau = I\alpha$	
Kinetic energy	$E = \frac{1}{2}mv^2$	$E_{\text{rot}} = \frac{1}{2}I\omega^2$	
Work	$W = \int \boldsymbol{F} \cdot d\boldsymbol{r}$	$W = \int \tau \, d\theta$	
Power	$P = \boldsymbol{F} \cdot \boldsymbol{v}$	$P = \tau\omega$	
Velocity	$v = v_i + at$	$\omega = \omega_i + \alpha t$	
Position	$z = z_i + v_i t + \frac{1}{2}at^2$	$\theta = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	

2.1.2 Torque

By multiplying Newton's second law for angular motion by r, we find

$$rF_t = mr^2\alpha = I\alpha, \tag{2.24}$$

so, we define the angular equivalent of force, called **torque**, as

$$\tau = rF_t \tag{2.25}$$

then, Newton's second law for angular motion becomes

$$\tau = I\alpha. \tag{2.26}$$

2.1.3 Motion for constant torque

For a constant torque τ the angular acceleration is

$$\alpha = \frac{\tau}{I} \tag{2.27}$$

and the angular velocity time t is

$$\omega(t) = \omega_i + \alpha t, \tag{2.28}$$

where ω_i is the initial angular velocity at $t = 0$. For the time dependent angle θ we find

$$\begin{aligned} \theta(t_f) &= \theta_i + \int_0^{t_f} \omega(t)\, dt \\ \theta_f &= \theta_i + \omega_i t_f + \frac{1}{2}\alpha t_f^2. \end{aligned} \tag{2.29}$$

Sometimes it is convenient to rewrite this expression to eliminate t. We define

$$\omega_f = \omega_i + \alpha t_f, \tag{2.30}$$

so $t_f = (\omega_f - \omega_i)/\alpha$ and

$$\begin{aligned} \theta_f &= \theta_i + \omega_i \frac{\omega_f - \omega_i}{\alpha} + \frac{1}{2}\alpha \left(\frac{\omega_f - \omega_i}{\alpha}\right)^2 \\ \theta_f &= \theta_i + \frac{\omega_f^2 - \omega_i^2}{2\alpha}. \end{aligned} \tag{2.31}$$

We can eliminate $\alpha = (\omega_f - \omega_i)/t_f$ instead, which gives

$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t, \tag{2.32}$$

where we effectively use the average angular velocity.

2.2 Moment arm

(See Figure 9.11 in Section 9.4 of Serway[1] in the section on torque). The angle between the **arm** ($\boldsymbol{r}$) and the force ($\boldsymbol{F}$) is equal to ϕ. The tangential force is $F_t = F \sin\theta$ and the torque is

$$\tau = rF \sin\theta = Fd, \tag{2.33}$$

where

$$d = r \sin\theta \tag{2.34}$$

is the **moment arm**. It it equal to the distance of the **line of action** to the rotation center, where the line of action is a line through the point $\boldsymbol{r}$, parallel to the force $\boldsymbol{F}$.

2.3 Work and power

The work done by an external tangential force is given by

$$dW = F_t\, ds = F_t r\, d\theta = \tau d\theta. \tag{2.35}$$

Only the tangential component of the force can do work, since there is no displacement in the radial direction. Integrating over θ gives the work done by the torque

$$W = \int \tau\, d\theta. \tag{2.36}$$

The **power** delivered by the force is

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega. \tag{2.37}$$

The work W done be the torque is again related to the difference in kinetic energy. First, for the torque

$$\sum \tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} \tag{2.38}$$

Integrating of the torque over θ

$$W = \int \tau d\theta = \int_{\omega_i}^{\omega_f} I\omega d\omega \tag{2.39}$$

so

$$W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2, \tag{2.40}$$

i.e., the work done by the torque in rotating the object around a fixed axis equals the change in the objects rotational energy.

2.4 Torque as vector

We will need the **cross product**, also called **vector product** of two vectors:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \tag{2.41}$$

In components this is

$$\boldsymbol{c} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_z \end{pmatrix}. \tag{2.42}$$

From this definition we find the $\boldsymbol{c} \perp \boldsymbol{a}$ and $\boldsymbol{c} \perp \boldsymbol{b}$

$$\boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{b} = 0. \tag{2.43}$$

For the length of the vector we have

$$c = ab \sin\phi, \tag{2.44}$$

where $a = |\boldsymbol{a}|$, $b = \boldsymbol{b}$, and $\phi = \angle(\boldsymbol{a}, \boldsymbol{b})$. Thus, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, i.e., $\phi = 0$, then $c = 0$.

We define **cross product** of the force and the arm as the the torque (vector):

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}. \tag{2.45}$$

It should be clear that for the **scalar torque** that we defined above, we have $\tau = |\boldsymbol{\tau}| = rF \sin\theta$. By defining the torque as vector, we do not only have the magnitude of the torque, but also the direction of the **rotation axis**. If we write the force as the sum of the tangential and radial components we find

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{r} \times (\boldsymbol{F}_t + \boldsymbol{F}_r) \\ &= \boldsymbol{r} \times \boldsymbol{F}_t + \boldsymbol{r} \times \boldsymbol{F}_r \\ &= \boldsymbol{r} \times \boldsymbol{F}_t,\end{aligned} \tag{2.46}$$

where the last step follows since $\boldsymbol{F}_r$ is parallel to $\boldsymbol{r}$.

The vectors $(\boldsymbol{F}_t, \boldsymbol{r}, \boldsymbol{\tau})$ are mutually orthogonal. Thus, if we know two of them, we can compute the other. If we normalize the vectors, $(\hat{\boldsymbol{F}}_t, \hat{\boldsymbol{r}}, \hat{\boldsymbol{\tau}})$, we can get the third vector as the cross product of the first two, also after cyclic permutation, e.g.,

$$\hat{\boldsymbol{F}}_t = \hat{\boldsymbol{\tau}} \times \hat{\boldsymbol{r}}. \tag{2.47}$$

For the magnitudes we can always use

$$F_t = \tau / r. \tag{2.48}$$

2.5 Rigid rotor

We can think of a rigid rotor as a molecule or a crystal consisting of n atoms, that are all rotating around the same axle with the same angular velocity ω. In reality, spinning a body will always deform it due to centrifugal forces, but if the angular velocity is sufficiently small, a system can be approximated as rigid. If the system consists of n atoms with masses m_i at a distance r_i to the rotation axis, the rotational kinetic energy is the sum of the contributions of all the atoms. Since by definition the angular velocity is the same for all atoms, the rotational kinetic energy can still be written as

$$T_{\text{rot}} = \frac{1}{2} I \omega^2, \tag{2.49}$$

where the moment of inertia is now the sum over all atoms

$$I = \sum_{i=1}^{n} m_i r_i^2. \tag{2.50}$$

The time dependent polar angle for a point/atom in the rigid body can be written as

$$\theta_i(t) = \theta_i(0) + \theta(t), \tag{2.51}$$

where $\theta_i(0)$ is the polar angle at $t = 0$. The angular velocity is the same for all points

$$\dot{\theta}_i = \dot{\theta}, \quad \text{for } i = 1, 2, \ldots, n, \tag{2.52}$$

and all other angular quantities, like the angular acceleration, and the rotational kinetic energy are given by the same expressions as before, but using the moment of inertia of the entire body. If more than one force is acting on the body, we can add the corresponding torques as vectors

$$\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i. \tag{2.53}$$

If we still consider a single rotation axis, with the tangential forces perpendicularly to it, then the torques are either parallel or anti-parallel to the rotation axis. When all torques are pointing in the same direction, we can simply add the magnitudes τ_i, but when a torque $\boldsymbol{\tau}_i$ is pointing in the opposite direction, we need to subtract its magnitude.

2.6 Computing moments of inertia for rigid bodies

For a macroscopic body the sum over all atoms can be converted to an integral over r. With the density $\rho(r)$ such that the mass in the interval $[r, r + dt]$ is given by

$$dm = \rho(r) dr \tag{2.54}$$

and the total mass is

$$M = \int_0^\infty \rho(r) dr \tag{2.55}$$

we get for the moment of inertia

$$I = \int_0^\infty \rho(r) r^2 dr. \tag{2.56}$$

For a **uniform solid cylinder** of radius R, and total mass M, rotating around its symmetry axis, the density $\rho(r)$ is proportional to r,

$$\rho(r) = fr \tag{2.57}$$

so for the total mass we have

$$M = \int_0^R \rho(r)\, dr = \frac{1}{2} f R^2 \tag{2.58}$$

i.e.,

$$f = 2\frac{M}{R^2}. \tag{2.59}$$

The moment of inertia is then

$$\begin{aligned} I_{\bullet} &= \int_0^R \rho(r) r^2\, dr \\ &= f \int_0^R r^3 dr = \frac{1}{4} f R^4 \\ &= \frac{1}{2} M R^2. \end{aligned} \tag{2.60}$$

For a thin cylindrical shell with diameter R, where all the mass is, at the edge, we have

$$\rho(r) = M\delta(r - R), \tag{2.61}$$

where we use the δ-functions defined by

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0), \tag{2.62}$$

for any reasonable function f. For the moment of inertia we get

$$I_{\bigcirc} = M \int_0^\infty \delta(r - R) r^2\, dr = M R^2. \tag{2.63}$$

For a long, thin **uniform rigid rod** of length L, rotating around its center-of-mass we have

$$\rho(r) = 2\frac{M}{L}, \quad \text{for } r < L/2, \tag{2.64}$$

such for the total mass we have

$$M = \int_0^\infty \rho(r)\, dr = M. \tag{2.65}$$

and for the moment of inertia

$$\begin{aligned} I_{\text{rod}} &= \int_0^{L/2} \rho(r) r^2\, dr \\ &= 2\frac{M}{L}\frac{1}{3}\left(\frac{L}{2}\right)^3 \\ &= \frac{1}{12} M L^2. \end{aligned} \tag{2.66}$$

If the long, thin **uniform rigid rod** of length L is rotating around an end-point, rather than the c.o.m. we have

$$\rho(r) = \frac{M}{L}. \tag{2.67}$$

Always check that the integral gives the total mass

$$\int_0^L \rho(r)\, dr = M. \tag{2.68}$$

The moment of inertia is

$$I = \int_0^L \rho(r) r^2 dr = \frac{1}{3} M L^2. \tag{2.69}$$

To compute the moment of inertia of a **rectangular beam** of width W, height H, and length L, it is more convenient to use Cartesian coordinates,

$$I = \int_0^L \int_{-w/2}^{w/2} \int_{-h/2}^{h/2} (x^2 + y^2) \rho dx\, dy\, dz. \tag{2.70}$$

For the mass of the beam we have

$$M = \rho W\,H\,L. \tag{2.71}$$

Using

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} \tag{2.72}$$

we find

$$\begin{aligned} I_\square &= \rho L \left(\frac{W^3}{12} + \frac{H^3}{12} \right) \\ &= M \frac{W^2 + H^2}{12}. \end{aligned} \tag{2.73}$$

We may compare this with a solid cylinder with the same mass and cross section

$$\pi R^2 = W H \tag{2.74}$$

so for a square beam with $W = H = R\sqrt{\pi}$ we get

$$I_\square = \frac{\pi}{3} I_\bigcirc. \tag{2.75}$$

2.7 Example: rotating Rod

(Serway[1], example 9.7 on uniform rigid rod). Horizontal rod, length L, mass M, with pivot on one end, starts rotating under gravity. Force of gravity, working on C.O.M.,

$$F = Mg. \tag{2.76}$$

Torque

$$\tau = FL/2. \tag{2.77}$$

Moment of inertia (see above)

$$\frac{1}{3} M L^2 \tag{2.78}$$

Angular acceleration

$$\alpha = \frac{\tau}{I} = \frac{FL/2}{ML^2/3} = \frac{3g}{2L}. \tag{2.79}$$

The (tangential) linear acceleration

$$a_t(r) = r\alpha = \frac{3g}{2L} r. \tag{2.80}$$

At the tip

$$a_t(L) = \frac{3}{2} g. \tag{2.81}$$

2.8 Angular momentum

We found the expression for linear momentum by integrating Newton's second law over time. If we do this for angular motion we get

$$\int_{t_i}^{t_f} \tau\, dt = \int_{t_i}^{t_f} I\alpha\, dt \tag{2.82}$$

On the rhs we get

$$\int_{t_i}^{t_f} I \frac{d\omega}{dt}\, dt = \int_{\omega_i}^{\omega_f} I d\omega = I\omega_f - I\omega_i. \tag{2.83}$$

So we define the **angular momentum** as

$$L = I\omega. \tag{2.84}$$

And we find that if the torque is zero, or if the integral over time of the torque is zero, the angular momentum is a conserved quantity. For a single particle in circular orbit with radius r we have

$$L = I\omega = mr^2 \frac{v_t}{r} = mrv_t = rp_t, \tag{2.85}$$

where v_t and p_t are the tangential velocities and momenta. As vector quantity we define the angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \tag{2.86}$$

which only picks up the tangential component of the linear momentum by the properties of a cross product. We find that $\boldsymbol{L}$ is parallel to the rotation axis.

2.9 Parallel-axis theorem

Above we derived expressions for the moment of inertia of several objects rotating around a symmetry axis. What if this object is rotating around another axis, parallel to the symmetry axis?

If we found the moment of inertia for rotation around an axis that contains the center of mass, I_{COM}, the moment of inertia for rotation around any axis that is parallel to this axis is given by

$$I = I_{\mathrm{COM}} + MD^2, \tag{2.87}$$

where D is the distance between the axes. We have already seen that the equation of motion of the C.O.M. of a system is decoupled from the relative motion within the system, so it may not come as a surprise that the moment of inertia is simply the sum of the moment of inertia of the object and the moment of inertia due to rotation of the C.O.M. around the parallel translated axis.

Here is a direct proof for a system consisting of point particles. We start with the z-axis to be the symmetry axis and the rotation axis, and we also assume that the C.O.M. is somewhere on this axis. The moment of inertia is given by

$$I_{\mathrm{COM}} = \sum_i m_i(x_i^2 + y_i^2), \tag{2.88}$$

and the C.O.M. (X, Y) is

$$X = \frac{\sum_i m_i x_i}{M} = 0 \tag{2.89}$$

$$Y = \frac{\sum_i m_i y_i}{M} = 0, \tag{2.90}$$

where $M = \sum_i m_i$.

Next, we translate the rotation axis to

$$\boldsymbol{d} = \begin{pmatrix} d_x \\ d_y \end{pmatrix} \tag{2.91}$$

so the coordinates of the particles become $(x_i + d_x, y_i + d_y)$ and the moment of inertia is

$$\begin{aligned} I &= \sum_i m_i[(x_i + d_x)^2 + (y_i + d_y)^2] \\ &= \sum_i m_i(x_i^2 + y_i^2) + \sum_i m_i(d_x^2 + d_y^2) \\ &\quad + 2\sum_i m_i(x_i d_x + y_i d_y). \end{aligned} \tag{2.92}$$

The third term is zero, since

$$\begin{aligned} \sum_i m_i(x_i d_x + y_i d_y) &= d_x \sum_i m_i x_i + d_y \sum_i m_i y_i \\ &= d_x MX + d_y MY = 0. \end{aligned} \tag{2.93}$$

so

$$I = I_{\mathrm{COM}} + MD^2 \tag{2.94}$$

with $D^2 = d_x^2 + d_y^2$.

Chapter 3

Rotation and translation of rigid bodies

We describe a rigid body as a system of point particles that are held in place relative to each other by internal forces. The rigid body is set in motion by external forces. A rigid body is an approximation which works well if the internal forces that keep the distances between the particles the same are large compared to the external forces, which in reality may distort the shape of the body at least a little bit. Here we show, using Newton's laws, that the motion can be found knowing the external forces, without having to solve for the internal forces.

In this chapter we will only consider problems where the velocities of the particles are all in the same two-dimensional plane. Note that this means that also the accelerations and forces are in this plane.

The force on particle (i) is

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{ext}} + \sum_j \boldsymbol{F}_{ij}, \tag{3.1}$$

where F_{ij} is the force on particle i as a result of the interaction with particle j. By Newton's 3rd law we have

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} \tag{3.2}$$

so, of course, $F_{ii} = 0$ because a particle does not exert a force on itself. We assume that the forces between the particles are **central**, i.e., they are parallel to the vector $\boldsymbol{r}_i - \boldsymbol{r}_j$. Examples would be particles connected with springs, Coulomb forces, or gravity.

The C.O.M. of the system is

$$\boldsymbol{X} = \frac{1}{M} \sum_i m_i \boldsymbol{r}_i. \tag{3.3}$$

Taking the time-derivative and multiplying with M gives the total linear momentum

$$\boldsymbol{P} = M\dot{\boldsymbol{X}} = \sum_i \boldsymbol{p}_i \tag{3.4}$$

and by Newton's 2nd law

$$\dot{\boldsymbol{P}} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{F}_i \tag{3.5}$$

$$= \sum_i \boldsymbol{F}_i^{\text{ext}} + \sum_i \sum_j \boldsymbol{F}_{ij}. \tag{3.6}$$

The second term, the contribution of the internal forces can we written as

$$\sum_i \sum_j \boldsymbol{F}_{ij} = \sum_{i<j} \boldsymbol{F}_{ij} + \sum_{i>j} \boldsymbol{F}_{ij}. \tag{3.7}$$

Here, the second term cancels the first, because

$$\sum_{i>j} \boldsymbol{F}_{ij} = \sum_{j>i} \boldsymbol{F}_{ji} = -\sum_{j>i} \boldsymbol{F}_{ij}, \tag{3.8}$$

keeping in mind that

$$\sum_{j>i} = \sum_{i<j}. \tag{3.9}$$

Thus, we have

$$\dot{\boldsymbol{P}} = M\ddot{\boldsymbol{X}} = \sum_i \boldsymbol{F}_i^{\text{ext}}. \tag{3.10}$$

Notice two things: **(i)** this result is also valid for a non-rigid body, since the derivation did not required the distances between the particles to be fixed.

(ii) We already found this result for gravity, but in that case the force of a particle $\boldsymbol{F}_i^{\text{ext}}$ is proportional to the mass of the particle m_i. Here we find that the equation of motion of the C.O.M. is controlled simply be the **sum of external forces**, regardless of the type of forces.

3.1 Rotation

The total torque is the sum of torques acting on the object

$$\begin{aligned} \boldsymbol{\tau} &\equiv \sum_i \boldsymbol{\tau}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{\text{ext}} + \sum_i \sum_j \boldsymbol{r}_i \times \boldsymbol{F}_{ij}. \end{aligned} \tag{3.11}$$

We can again use Newton's 3rd law to show that the contribution of the internal forces is zero

$$\sum_{ij} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = \sum_{i<j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \sum_{i>j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}. \tag{3.12}$$

For the second term we have

$$\sum_{i>j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = -\sum_{i>j} \boldsymbol{r}_j \times \boldsymbol{F}_{ji} = -\sum_{j>i} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} \tag{3.13}$$

so the total internal torque is

$$\sum_{i<j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = 0, \tag{3.14}$$

where we assumed the force $\boldsymbol{F}_{ij}$ is parallel to $\boldsymbol{r}_i - \boldsymbol{r}_j$.

The result is actually more general: it does not require central forces, but is valid when space is isotropic, i.e., there is no preferred orientation of the frame used to write down the equations of motion.

If we translate the rotation axis by $\boldsymbol{d}$ we find

$$\begin{aligned} \boldsymbol{\tau}' &= \sum_i (\boldsymbol{r}_i - \boldsymbol{d}) \times \boldsymbol{F}_i && (3.15) \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i - \sum_i \boldsymbol{d} \times \boldsymbol{F}_i && (3.16) \\ &= \boldsymbol{\tau} - \boldsymbol{d} \times \sum_i \boldsymbol{F}_i, && (3.17) \end{aligned}$$

so the result still only depends on external forces

$$\boldsymbol{\tau}' = \boldsymbol{\tau} - \boldsymbol{d} \times \boldsymbol{F}^{\text{ext}}. \tag{3.18}$$

3.2 Static equilibrium

An rigid object that is not moving will remain in place if the sum of external forces as well as the sum of the torques due to external forces is zero. Because of Eq. (3.15) the torque may be computed for any rotation axis.

3.3 Cylinder rolling down slope

(See Fig 10.9 in section 10.2 on Rolling Motion of a Rigid Object in Serway[1]). Consider a cylinder with radius R, total mass M, which is at rest at time $t = 0$ and starts rolling down a slope with angle θ with the horizontal plane. The questions we want to answer are: does the time it takes to roll down the slope depend on the mass (M)? Does the time depend on the radius (R)? Is there a difference between a solid cylinder and a thin cylindrical shell?

First, we must determine all external forces on the cylinder. For each force we must know its magnitude, its direction, and the point on the object where it acts:

- The force that sets the cylinder in motion is gravity
$$\boldsymbol{F}_g = -mg\hat{\boldsymbol{e}}_z, \tag{3.19}$$
where $\hat{\boldsymbol{e}}_z$ is a unit vector pointing up. This force is acting on the C.O.M. of the cylinder, i.e., on the symmetry axis.

- The cylinder is not in free-fall, so there must be a **normal force** perpendicular to the surface, $\boldsymbol{F}_N$. It acts on the point of the cylinder that touches the surface.

If there is no friction at all between the cylinder and the plane, the cylinder would slide down the slope without starting to rotate. So, if the cylinder doesn't slide, there must be a friction force ($\boldsymbol{F}_f$) acting on the point where the cylinder touches the surface. Friction forces are parallel to the surface, i.e., along the slope. (When solving such a problem, always make a drawing with the object and the forces).

For convenience, we define unit vectors parallel ($\hat{\boldsymbol{e}}_\parallel$) and perpendicular ($\hat{\boldsymbol{e}}_\perp$) to the surface of the slope, and we write decompose the force due to gravity in parallel and perpendicular component, where $\hat{\mathbf{e}}_\parallel$ is assumed to point downwards

$$\begin{aligned}\boldsymbol{F}_g &= \boldsymbol{F}_{g,\perp} + \boldsymbol{F}_{g,\parallel} \\ &= mg\cos\theta\,\hat{\boldsymbol{e}}_\perp + mg\sin\theta\,\hat{\boldsymbol{e}}_\parallel\end{aligned} \tag{3.20}$$

The **normal force** is a **reactive force** that must cancel the perpendicular component, so

$$\boldsymbol{F}_N = -\boldsymbol{F}_{g,\perp} \tag{3.21}$$

The motion of the C.O.M. of the rigid body is controlled by the sum of all external forces, which in this case is

$$\boldsymbol{F}^{\text{ext}} = \boldsymbol{F}_{g,\parallel} + \boldsymbol{F}_f + \boldsymbol{N} + \boldsymbol{F}_{g,\perp}. \tag{3.22}$$

The last two terms cancel, and for the friction we have

$$\boldsymbol{F}_f = -F_f\hat{\boldsymbol{e}}_\parallel. \tag{3.23}$$

Thus, we have

$$\boldsymbol{F}^{\text{ext}} = (Mg\sin\theta - F_f)\hat{\boldsymbol{e}}_\parallel. \tag{3.24}$$

We cannot yet solve the equation of motion of the C.O.M., since we do not yet know F_f.

The rotational motion is controlled by the sum of all external torques. These torques must be determined with respect to the axis running through the C.O.M. of the cylinder. Gravity is acting on the C.O.M., so it does not give a torque. The torque due to the **normal force** is

$$\boldsymbol{\tau}_N = \boldsymbol{R} \times \boldsymbol{F}_N = \boldsymbol{0}, \tag{3.25}$$

where $\boldsymbol{R}$ is vector pointing from the C.O.M. of the cylinder to the point where the cylinder touches the surface. The result is zero, since the vectors in the cross product are parallel. The only force that contributes to the torque is the friction force,

$$\boldsymbol{\tau}_f = \boldsymbol{R} \times \boldsymbol{F}_f. \tag{3.26}$$

Since the vectors are perpendicular, we have for the magnitude

$$\tau_f = RF_f. \tag{3.27}$$

The equations of motion for translation and rotation are

$$Mg\sin\theta - F_f = Ma \tag{3.28}$$
$$RF_f = I\alpha, \tag{3.29}$$

where I is the moment of inertia of the cylinder. We have, however, three unknowns: the linear acceleration a, the angular acceleration α, and the friction force F_f. We have one extra equation though, because the linear and angular acceleration are related:

Rotation of the cylinder over an angle θ makes the contact point and also the C.O.M. travel over a distance

$$s = R\theta. \tag{3.30}$$

With $a = \ddot{s}$ and $\alpha = \ddot{\theta}$ this gives

$$a = R\alpha. \tag{3.31}$$

We can now relate the friction to the linear acceleration

$$F_f = \frac{Ia}{R^2} \tag{3.32}$$

and Eq. (3.28) becomes

$$g\sin\theta - \frac{I}{MR^2}a = a \tag{3.33}$$

so

$$a = \frac{g\sin\theta}{1 + I/(MR^2)}. \tag{3.34}$$

Thus the acceleration is constant. If the slope get steeper, $\sin\theta$ get bigger and the acceleration increases. For fixed slope, the acceleration depend on the ration of the moment of inertial I and MR^2. For a thin cylindrical shell, all the mass is at a distance R and $I = MR^2$, so the ratio is one, and the acceleration is half of what it would be for frictionless sliding. When the mass is, on average, closer to the rotation axis, as in the case of a solid cylinder, we have $I/MR^2 = \frac{1}{2}$, so the acceleration is larger.

3.4 Rolling cylinder, conservation of energy

Since the friction does not produce heat we can also use energy conservation to solve the same problem. The translational kinetic energy is

$$T_{\text{trans}} = \frac{1}{2}Mv^2, \tag{3.35}$$

where v is the linear velocity of the C.O.M. of the cylinder. The rotational kinetic energy is

$$T_{\text{rot}} = \frac{1}{2}I\omega^2. \tag{3.36}$$

If the C.O.M. is at a height h, the potential energy is

$$V(h) = Mgh. \tag{3.37}$$

Assuming the cylinder starts at height $h = h_i$ and rolls down to $h = 0$, energy conservation gives

$$Mgh = \frac{1}{2}M{v_f}^2 + \frac{1}{2}I\omega_f^2, \tag{3.38}$$

where we assumed that $v_i = \omega_i = 0$. Again, we can use Eq. (3.30) to relate translational and rotational motion. The first derivative with respect to time gives

$$v_f = R\omega_f \tag{3.39}$$

so we find

$$Mgh = \frac{1}{2}M{v_f}^2 + \frac{1}{2}\frac{I}{R^2}v_f^2, \tag{3.40}$$

i.e.,

$$\frac{1}{2}v_f^2 = \frac{gh}{1 + I/(MR^2)}. \tag{3.41}$$

For a given height h, the final velocity is controlled by the ratio $I/(MR^2)$, just like we found for the acceleration.

The vertical distance h is related to the distance traveled along the surface, s by

$$h = s \sin \theta, \tag{3.42}$$

so the time derivative is related to v_f

$$\dot{h} = v_f \sin \theta. \tag{3.43}$$

Taking the time-derivative of Eq. (3.41) gives

$$v_f a = \frac{g v_f \sin \theta}{1 + I/(MR^2)} \tag{3.44}$$

and v_f cancels, so we get, as before

$$a = \frac{g \sin \theta}{1 + I/(MR^2)}. \tag{3.45}$$

Chapter 4

Angular momentum

Angular momentum of a particle is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}. \tag{4.1}$$

Thus, angular momentum arises in the case of circular motion around an axis through the origin, perpendicular to the plane containing $\boldsymbol{r}$ and $\boldsymbol{p}$. However, a particle can also have nonzero angular momentum due to linear motion: it only requires that $\boldsymbol{r}$ and $\boldsymbol{p}$ are nonzero and not parallel:

$$L = |\boldsymbol{L}| = rp\sin\theta, \tag{4.2}$$

with θ the angle between $\boldsymbol{r}$ and $\boldsymbol{p}$.

The time-derivative of $\boldsymbol{L}$ is

$$\begin{aligned} \dot{\boldsymbol{L}} &= \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}}. && (4.3)\\ &= \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{r} \times \boldsymbol{F} && (4.4)\\ &= \boldsymbol{0} + \boldsymbol{\tau}. && (4.5) \end{aligned}$$

For a system of particles, the total angular momentum is the sum

$$\boldsymbol{L} = \sum \boldsymbol{r}_i \times \boldsymbol{p}_i \tag{4.6}$$

and for the time derivative we get

$$\dot{\boldsymbol{L}} = \sum \boldsymbol{\tau}^{\text{ext}}, \tag{4.7}$$

where we only sum over external torques, since the contributions from the internal torques cancel. This last equation also applies to rigid bodies. When the external torques are zero, we find that $\boldsymbol{L}$ is a new **conserved quantity**.

For circular motion, with $\boldsymbol{r}$ perpendicular to the rotation axis we have

$$v = r\omega \tag{4.8}$$

and

$$L = mr^2\omega = I\omega \tag{4.9}$$

For the rotational kinetic energy we have

$$T_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}. \tag{4.10}$$

4.1 Non-isolated system

We already solved the rolling cylinder problem using forces and using energy conservation. We can also solve it using angular momentum, which is not converged, since there is a torque acting on the cylinder.

The angular momentum relative to the contact point is

$$L = (I + MR^2)\omega, \tag{4.11}$$

where I is the moment of inertia for rotation around the symmetry axis of the cylinder, and the MR^2 contribution to the moment of inertia follows from the parallel axis theorem.

The torque now comes from the parallel component of the force of gravity only, since the friction force does not give a torque on the contact point. Thus, we have

$$\dot{L} = \tau \quad (4.12)$$

$$(I + MR^2)\alpha = RF_\parallel = RMg\sin\theta, \quad (4.13)$$

and with $a = R\alpha$ we get

$$(I + MR^2)\frac{a}{R} = RMg\sin\theta \quad (4.14)$$

and for the third time we get

$$a = \frac{g\sin\theta}{1 + I/(MR^2)}. \quad (4.15)$$

Another example is mass m_1 sliding frictionless horizontally, being pulled by a rope which is wrapped around a pulley and holding a mass m_2 which is pulled down by gravity. The radius of the pulley is r. The pulley has mass m_p, and it is assumed all the mass is at the rim, i.e., $I = m_p r^2$. The angular momentum of the system has three sources of angular momentum with respect to the rotation axis of the pulley:

$$L = rm_1v + rm_2v + rm_pv \quad (4.16)$$

The external torque is $\tau = rF_g$, so

$$\dot{L} = \tau \quad (4.17)$$

$$r(m_1 + m_2 + m_p)a = rF_g \quad (4.18)$$

If you also want to know the internal forces you get three equations with three unknowns

$$F_1 = m_1 a \quad (4.19)$$

$$F_g - F_2 = m_2 a \quad (4.20)$$

The forces acting on the pulley are F_1 (horizontal) and F_2 (vertical), so the torque is $\tau = r(F_2 - F_1)$ and

$$\tau = I\alpha = Ia/r \quad (4.21)$$

so

$$F_2 - F_1 = \frac{I}{r^2}a. \quad (4.22)$$

Adding Eqs. (4.19), (4.20), and (4.22) gives again Eq. (4.18). After finding the acceleration a, we can use the Eqs. (4.19) and (4.20) to compute the internal forces.

4.2 Isolated systems, conservation of angular momentum

Without external torques, the total angular momentum $\boldsymbol{L}$ is conserved. Since $\boldsymbol{L}$ is a vector, this means all three components are conserved.

Since $L = I\omega$ is conserved, ω will increase if the moment of inertia gets smaller. Examples are a figure skater doing a piroutte pulling in her arms, which makes her spin faster, or a person walking to the center of a "Merry-Go-Round" while it rotates. In both cases there are no external torques, but the angular velocity changes.

Even though there are not external forces, the rotational energy is not conserved.

As an example, consider a rotating platform (Fig 10.19 in Serway[1]), with moment of inertia I, and a person with mass m at a distance r from the center. The initial angular momentum is

$$L_i = I\omega_i + mr^2\omega_i \quad (4.23)$$

If the person walks to the center (and we neglect the moment of inertia of the person), we have

$$L_f = I\omega_f. \quad (4.24)$$

Conservation of angular momentum $L_i = L_f$ then gives

$$\omega_f = \omega_i \frac{I + mr^2}{I}. \tag{4.25}$$

We may also write this result as

$$\omega_f = \omega_i \frac{I_i}{I_f}, \tag{4.26}$$

with $I_i = I + mr^2$ and $I_f = I$.

For the rotation energy we find

$$\frac{E_f}{E_i} = \frac{L^2/2I_f}{L^2/2I_i} = \frac{I_i}{I_f}, \tag{4.27}$$

so clearly, this is not conserved. The difference $E_f - E_i$ is the work done by the person walking to the center.

4.3 Spinning tops and gyroscopes

A top is a rigid body with cylinder symmetry. It can spin around its symmetry axis with the tip resting on the pivot point (Serway[1], 10.21). The C.O.M. is on the symmetry axis. If the symmetry axis makes an angle θ with the vertical, the C.O.M. is not above the pivot point, so gravity will exert a torque. With the pivot point in the origin, and $\boldsymbol{r}$ the C.O.M., the torque exerted by gravity is

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}_g = M\boldsymbol{r} \times \boldsymbol{g}. \tag{4.28}$$

Thus $\boldsymbol{\tau}$ is in the horizontal plane, and perpendicular to the symmetry axis, $\boldsymbol{r}$, as well as the vertical axis $\boldsymbol{g}$. As a result the plane containing the vertical and the symmetry axis will start rotating around the vertical. Since $\dot{L}$ is perpendicular to $\boldsymbol{L}$ it will make $\boldsymbol{L}$ rotate with angular velocity

$$\omega_L = \frac{|\dot{\boldsymbol{L}}|}{|\boldsymbol{L}|} = \frac{\tau}{L} \tag{4.29}$$

Now we make the assumption that $\boldsymbol{L}$ remains parallel to $\boldsymbol{r}$. This would be exact if the only contribution to the angular momentum is the spinning of the top. If the top spins fast enough, this becomes a good approximation. Thus, we assume that $\boldsymbol{r}$ also has an angular velocity ω_L. Its linear velocity will be $v = r\omega_L$. For the angular velocity for rotation around the vertical we have

$$v = r\sin(\theta)\omega_p \tag{4.30}$$

so

$$\omega_p = \frac{v}{r\sin\theta} = \frac{r\omega_L}{r\sin\theta} = \frac{Mrg}{I\omega}. \tag{4.31}$$

This result is only valid if $\omega \gg \omega_p$.

Chapter 5

Gravity and Kepler's laws

Masses m_1 and m_2 attract each other with a force given by

$$F = G\frac{m_1 m_2}{r^2}, \tag{5.1}$$

where r is the distance between the particles and the **universal gravitational constant** is given by

$$G = 6.673 \times 10^{-11}\text{Nm}^2/\text{kg}^2. \tag{5.2}$$

In vector form, the force on particle 1 due to particle 2:

$$\boldsymbol{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}_{12} \tag{5.3}$$

where the direction is given by the unit vector

$$\hat{\boldsymbol{r}}_{12} = \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{r}. \tag{5.4}$$

The attraction due to the mass of a spherical object is the same as if the mass were concentrated in the center, e.g., a mass just above the surface of the Earth, so the free fall acceleration is

$$\begin{aligned} g &= \frac{F}{m} = G\frac{M_E}{R_E^2} \\ &= 6.673 \times 10^{-11}\frac{5.98 \times 10^{24}}{(6.37 \times 10^6)^2} = \\ &= 9.8 \text{ m/s}^2. \end{aligned} \tag{5.5}$$

In vector form this is called the **gravitational field**

$$\boldsymbol{g} = \frac{\boldsymbol{F}_g}{m} = -\frac{GM_E}{r^2}\,\hat{\boldsymbol{r}}, \tag{5.6}$$

with $\hat{\boldsymbol{r}}$ a unit vector pointing up. For any motion on the surface of the Earth (modeled as a perfect sphere) no work due to gravity is done, since

$$dW = \boldsymbol{F} \cdot d\boldsymbol{s} = 0. \tag{5.7}$$

The work done by gravity due to gravity for vertical motion from a distance r_i from the center of the Earth to a distance r_f from the center of the Earth is

$$\begin{aligned} W &= \int_{r_i}^{r_f} F_g(r)dr = -\int_{r_i}^{r_f}\frac{GM_E m}{r^2} \\ &= GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) \equiv V_g(r_i) - V_g(r_f). \end{aligned} \tag{5.8}$$

So the force of gravity is a **conservative force**. If we take the potential to be zero of $r_f = \infty$ we get

$$V_g(r) = -\frac{GM_E m}{r}. \tag{5.9}$$

As a check, we can compute the force

$$\boldsymbol{F}_g = -\boldsymbol{\nabla} V_g(r). \tag{5.10}$$

For the first component of the gradient we get

$$\frac{\partial V_g(r)}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial V_g(r)}{\partial r}, \tag{5.11}$$

with

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}} \\ &= 2x\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{x}{r}.\end{aligned} \tag{5.12}$$

If we do the same for the y and z coordinates we get

$$\boldsymbol{F}_g = -(\boldsymbol{\nabla} r)\frac{\partial V_g(r)}{\partial r} = -\hat{\boldsymbol{r}}\frac{GM_E m}{r^2}. \tag{5.13}$$

Dividing this equation by m reproduces the gravitational field [Eq. (5.6)].

The **weight** of an object is defined as the force of gravity on the object. Note that the use of the phrase "experiencing **weightlessness** in free fall, or in orbit" is not consistent with this definition, since the force of gravity is not zero if you fall. As an example: an astronaut with a mass of $m = 75$ kg in the ISS at a height of $h = 350$ km above the Earth surface has a weight

$$w = \frac{GM_E m}{(R_E+h)^2} = mg\frac{R_E^2}{(R_E+h)^2} = 660\,\mathrm{N}. \tag{5.14}$$

5.1 Orbital motion

Consider a particle with mass m in orbit around a mass M, with $M \gg m$, so the heavy mass can be approximated as stationary. The total energy is

$$E = T + V_g = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{5.15}$$

If the total energy $E < 0$ the system is **bound**, since r can be at most

$$r \leq \frac{GMm}{-E}. \tag{5.16}$$

For a **circular orbit** the radial acceleration [Eq. (2.17)] must be equal to the gravitational force divided by the mass

$$a = \frac{v^2}{r} = \frac{GM}{r^2}. \tag{5.17}$$

Thus, for the kinetic energy we get

$$T = \frac{1}{2}mv^2 = \frac{GMm}{2r} = -V_g/2 \tag{5.18}$$

and for the **orbital energy** we get

$$E = T + V_g = V_g/2. \tag{5.19}$$

With this expression, together with Eq. (5.9) one can easily calculate the energy required to go from a circular orbital with radius r_i to an orbital with radius r_f.

We can also use Eq. (5.19) to compute the **escape speed**. For $E = 0$ we get

$$T = -V_g \tag{5.20}$$

so

$$\frac{1}{2}mv^2 = \frac{GM_E m}{r} \tag{5.21}$$

and

$$v = \sqrt{\frac{2GM}{r}}. \tag{5.22}$$

For "**escaping the Earth gravitational field**" this gives

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM_E}{r_E}} = 1.12\times 10^4\ \mathrm{m/s} \approx 40\,000\,\mathrm{km/h}. \tag{5.23}$$

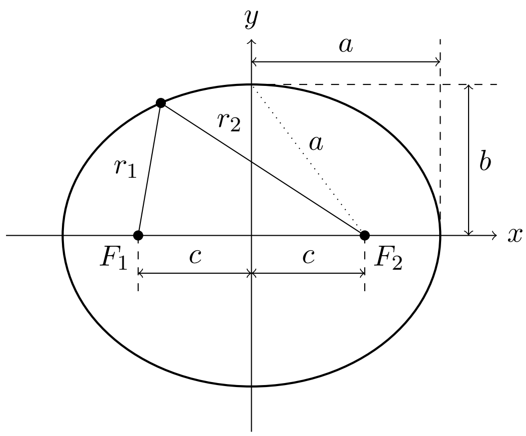

Figure 5.1: An ellipse with **semi-major axis** a, **semi-minor axis** b, **eccentricity** c, and **focal points** F_1 and F_2.

5.2 Kepler's laws

Tycho Brahe (Denmark, 1546–1601) observed and recorded the motion of planets and stars using **quadrants** and **sextants**, but no telescopes, since they were only invented in 1608 (by Hans Lippershey, Netherlands). Johannes Kepler (Germany, 1571-1630) was his assistant and spend 16 years to analyze the data and derive his laws.

Note that Newton published his laws much later, in 1687. Kepler's laws can be derived from Newton's laws. It is not easy though, since the explicit expression for the position of planets as a function of time cannot be expressed in elementary mathematical functions.

5.2.1 Kepler's first law

"The orbits of planets are ellipses, with the Sun in a **focal point** *of the ellipse."*

The points (x, y) on an ellipse satisfy the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \tag{5.24}$$

where a and b are the **semimajor** and **semiminor** axes, resp. [Fig. 5.1]. The focal points F_1 and F_2 have coordinates $(-c, 0)$ and $(c, 0)$, with

$$a^2 = b^2 + c^2, \tag{5.25}$$

i.e., such that the dotted line in Fig. (5.1) has length a.

The sum of the distances of point $(0, a)$ to the focal points is

$$(a + c) + (a - c) = 2a \tag{5.26}$$

which is the same as the sum of of the distances of point $(0, b)$ to the focal points. It turns out that the sum of the distances from any point on the circle to the focal points is the same

$$r_1 + r_2 = 2a. \tag{5.27}$$

On Wikipedia you can find a hint on how to prove this.

The **eccentricity** of an ellipse, e, is defined as

$$e = c/a. \tag{5.28}$$

For a circle $a = b$, so $c = 0$ and also $e = 0$. The eccentricity can be at most $e = 1$ (why?).

The eccentricity of the Earth orbit is $e_{\text{Earth}} = 0.017$ and for other planets it is also small. For Comets it can be much larger, e.g., for Comet Halley it is 0.97.

5.2.2 Kepler's second law

"The sun-planet radius vector sweeps out equal areas in equal time intervals". This law follows from conservation of angular momentum. From

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{5.29}$$

we get

$$\boldsymbol{r} \times \boldsymbol{v} = \frac{1}{m}\boldsymbol{L} \tag{5.30}$$

Thus,

$$rv\sin\theta = rv_\perp = L/m, \tag{5.31}$$

where $v_\perp$ is the component of the velocity that is perpendicular to $\boldsymbol{r}$. For a small time interval dt we have of the area that is "swept out"

$$dA = \frac{1}{2}rv_\perp dt = \frac{1}{2}\frac{L}{m}dt, \tag{5.32}$$

so

$$\dot{A} = \frac{1}{2}\frac{L}{m}. \tag{5.33}$$

Integrating over time of dA/dt from t_i to t_f gives

$$A = \int_{t_i}^{t_f}\frac{dA}{dt}dt = \frac{1}{2}\frac{L}{m}(t_f - t_i), \tag{5.34}$$

which is Kepler's second law.

Note that this law does not depend on the gravitational potential, it only requires conservation of angular momentum.

5.2.3 Kepler's third law

"The square of the orbital period (T_{orb}) of any planet is proportional to the cube of the semimajor axis of the elliptical orbit"

$$T_{\mathrm{orb}}^2 \propto a^3. \tag{5.35}$$

For a circular orbit we have

$$T_{\mathrm{orb}} = 2\pi r/v, \tag{5.36}$$

so with Eq. (5.18) we have

$$T_{\mathrm{orb}}^2 = \frac{4\pi^2}{GM}r^3. \tag{5.37}$$

For an elliptic orbit, the orbital period is found by replacing r by the semimajor axis a in this equation.

We did not yet prove that in general the orbits are elliptic, but if we accept Kepler's first law, we can find the orbital period by using energy and angular momentum conservation.

For the **aphelion** the distance of the planet to the sun is

$$r_a = a + c \tag{5.38}$$

and for the **perihelion** it is

$$r_p = a - c. \tag{5.39}$$

We will derive the orbital period using Kepler's 2nd law

$$T^2 = \frac{A^2}{\dot{A}^2}, \tag{5.40}$$

where A is the area of an ellipse

$$A = \pi ab. \tag{5.41}$$

Both in the aphelion and in the perihelion the velocity of the planet is perpendicular to the vector $\boldsymbol{r}$, so **conservation of angular momentum** gives us

$$r_a v_a = r_p v_p, \tag{5.42}$$

and we can use

$$\dot{A} = \frac{1}{2}r_a v_a. \tag{5.43}$$

The sum of kinetic and potential energy in the aphelion and perihelion must be the same by **conservation of energy**, which gives

$$\frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} \tag{5.44}$$

so

$$v_a^2 - v_p^2 = 2GM\left(\frac{1}{r_a} - \frac{1}{r_p}\right). \tag{5.45}$$

We can relate v_p to v_a with Eq. (5.42)

$$v_p = v_a\frac{r_a}{r_p} \tag{5.46}$$

so

$$v_a^2\left(1-\frac{r_a^2}{r_p^2}\right)=2GM\frac{r_p-r_a}{r_ar_p}. \qquad (5.47)$$

Using

$$\left(1-\frac{r_a^2}{r_p^2}\right)=\frac{1}{r_p^2}(r_p+r_a)(r_p-r_a) \qquad (5.48)$$

this simplifies to

$$v_a^2\frac{r_a+r_p}{r_p}=2GM\frac{1}{r_a}, \qquad (5.49)$$

i.e.,

$$r_a^2v_a^2=2GM\frac{r_ar_p}{r_a+r_p} \qquad (5.50)$$

From Eqs. we have

$$r_a+r_p=2a \qquad (5.51)$$

$$r_ar_p=a^2-c^2=a^2(1-e^2) \qquad (5.52)$$

which gives

$$r_a^2v_a^2=GMa(1-e^2) \qquad (5.53)$$

and so

$$T^2=\frac{A^2}{\dot{A}^2}=\frac{4\pi^2}{GM}a^3, \qquad (5.54)$$

which is the formula for a circular orbit with radius equal to the semimajor axis.

5.3 Elliptic orbits, Newton's laws

To derive Kepler's first and third law's several tricks are needed. Here we see how far we get with a "naive" use of Newton's second law. We adapt the approach for circular motion of Chap. 2 by making $r=r(t)$ time-dependent,

$$\boldsymbol{r}(t)=r(t)\hat{\boldsymbol{r}}(t). \qquad (5.55)$$

For the time derivatives of the directions we already found

$$\frac{\partial}{\partial t}\hat{\boldsymbol{r}}(t)=\omega\hat{\boldsymbol{r}}_\perp(t) \qquad (5.56)$$

$$\frac{\partial}{\partial t}\hat{\boldsymbol{r}}_\perp(t)=-\omega\hat{\boldsymbol{r}}(t) \qquad (5.57)$$

so

$$\dot{\boldsymbol{r}}=\dot{r}\hat{\boldsymbol{r}}+r\omega\hat{\boldsymbol{r}}_\perp \qquad (5.58)$$

and

$$\ddot{\boldsymbol{r}}=\ddot{r}\hat{\boldsymbol{r}}+\dot{r}\omega\hat{\boldsymbol{r}}_\perp+\dot{r}\omega\hat{\boldsymbol{r}}_\perp+r\alpha\hat{\boldsymbol{r}}_\perp-r\omega^2\hat{\boldsymbol{r}}. \qquad (5.59)$$

From Newton's second law we have

$$\ddot{\boldsymbol{r}}=-\frac{MG}{r^2}\hat{\boldsymbol{r}}. \qquad (5.60)$$

This shows that the perpendicular component must be zero, so

$$a_\perp=2\dot{r}\omega+r\alpha=0. \qquad (5.61)$$

This turns out to be conservation of angular momentum:

$$\dot{L}=\frac{\partial}{\partial t}I\omega=\frac{\partial}{\partial t}mr^2\omega= \\ =m(2r\dot{r}\omega+r^2\alpha)=0. \qquad (5.62)$$

Dividing this equation by mr gives Eq. (5.61).

For the parallel component we get

$$\ddot{r}-r\omega^2=-\frac{MG}{r^2}. \qquad (5.63)$$

Thus, if we require r to be constant, $\ddot{r}=0$, we get

$$r\omega^2=\frac{MG}{r^2}. \qquad (5.64)$$

With

$$\omega=\frac{2\pi}{T_{\text{orb}}}, \qquad (5.65)$$

this gives Kepler's 3rd law for circular motion.

For noncircular motion we must solve Eq. (5.63). It may seem we have two unknowns, $r(t)$ and $\omega(t)$, but these are related through conservation of angular momentum,

$$\omega = \frac{L}{mr^2} \tag{5.66}$$

and we can rewrite Eq. (5.63) as

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{mMG}{r^2}. \tag{5.67}$$

This is Newton's second law for one-dimensional motion along r with a radial potential

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V_g(r) \equiv \frac{A}{r^2} - \frac{B}{r}. \tag{5.68}$$

with

$$A = \frac{L^2}{2m} \tag{5.69}$$

$$B = GMm. \tag{5.70}$$

By ingenious substitutions, this equation can be solved. Qualitatively, we see a potential that is repulsive for small r, where the centrifugal term is the largest, and a potential that is attractive at large r, where gravity wins. We must get a circular orbit at the minimum of $V_{\text{eff}}(r_0)$,

$$V_{\text{eff}}(r_0)' = -2\frac{A}{r_0^3} + \frac{B}{r_0^2} = 0 \tag{5.71}$$

so

$$r_0 = \frac{2A}{B} \tag{5.72}$$

which gives

$$\frac{v^2}{r_0} = \frac{MG}{r_0^2}, \tag{5.73}$$

which is indeed the condition for circular orbits.

We now consider an orbit that is close to circular

$$r(t) = r_0 + x(t), \tag{5.74}$$

where $x(t)$ is small compared to r. We approximate the effective potential as an harmonic oscillator around $r = r_0$,

$$V_{\text{eff}}(r_0 + x) = \frac{1}{2}kx^2 \tag{5.75}$$

with

$$k = V''_{\text{eff}}(r_0) = \frac{6A}{r_0^4} - \frac{2B}{r_0^3} = \frac{2A}{r_0^4}, \tag{5.76}$$

where we used Eq. (5.72). The solution of the harmonic oscillator problem with mass m and force constant k is

$$x(t) = c\cos(\omega_0 t) \tag{5.77}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2L^2}{2m^2r_0^4}} = \frac{L}{mr_0^2}. \tag{5.78}$$

Thus, ω_0 is the same as the angular velocity for a circular orbit with radius r_0, since $L = I\omega$, so

$$\omega = L/I = L/mr_0^2 = \omega_0. \tag{5.79}$$

Thus after a complete orbit with $T = 2\pi/\omega$, the amplitude $x(t)$ has also made a complete oscillation and the orbit is closed.

Note that if we solved $r(t)$ we can get $\omega(t)$ from Kepler's second law. For small amplitude $x(t)$ the orbital period does not change in first order, since

$$\begin{aligned}\omega(t) = \frac{L}{mr(t)^2} &= \frac{L}{mr_0^2}[1 - 2\frac{x(t)}{r_0} + \ldots] \\ &= \omega_0[1 - 2\frac{x(t)}{r_0} + \ldots].\end{aligned} \tag{5.80}$$

We can find $\phi(t)$ by integrating $\omega(t)$. After integrating from $t = 0$ to $t = T$ we get

$$\phi(T) = \int_0^T \omega(t)dt = \omega_0 T = 2\pi. \tag{5.81}$$

According to Kepler, with the exact $x(t)$ it holds to any order.

Chapter 6

Fluid statics

A fluid is a liquid or a gas. In this chapter we consider fluids that are not moving. The density of a fluid is defined as mass m per volume V,

$$\rho = \frac{m}{V}. \quad (6.1)$$

Pressure (P) is force (F) divided by area

$$P = \frac{F}{A}. \quad (6.2)$$

6.1 Pascal's law

Pascal's law for pressure $P(h)$ as function of depth (h)

$$P(h) = P_0 + \rho g h, \quad (6.3)$$

where P_0 is the pressure at the surface. The force due to the pressure is perpendicular to the wall of the container.

For a vertical cylinder with height h and area A, and hence volume $V = Ah$, filled with a fluid with density ρ, the difference between upward force at the bottom and downward force at the top is

$$\begin{aligned} F_{\text{up}} - F_{\text{down}} &= AP - AP_0 \\ &= A\rho g h \\ &= V\rho g = mg \\ &= F_g, \end{aligned} \quad (6.4)$$

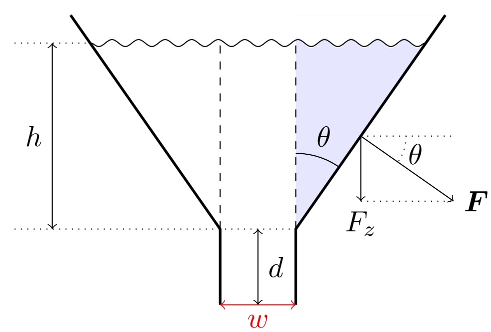

Figure 6.1: Pascal's law.

i.e., it is the force of gravity on the fluid.

In a mercury barometer there is a vacuum above the liquid surface so the pressure $P_0 = 0$, and the atmospheric pressure is measured

$$P_{\text{atm}} = \rho_{\text{Hg}} g h, \quad (6.5)$$

with the density of mercury $\rho_{\text{Hg}} = 13.6 \times 10^3$ kg/m^3. With one atmosphere defined as 101 kPa, this gives $h = 76$ cm.

Figure 6.1 shows the profile of a container with length L. The width at the bottom is w, so the area of the bottom of the container is $A = wL$. The pressure at a depth d is $P = g\rho(h + d)$, so the downward force is

$$F = PA = g\rho(h + d)wL = g\rho V = gM, \quad (6.6)$$

is the the force of gravity on the water directly above the bottom of the container. The shape of the container does not matter for Pascal's law.

Here we show that the weight of the water in the triangular part of the container (the blue part) is supported by the slanted part of the wall directly below the water. The volume is

$$V_1 = \frac{1}{2} d^2 L \tan\theta \tag{6.7}$$

so the downward force is due to gravity is

$$F_g = g\rho V_1 = \frac{1}{2} g\rho d^2 L \tan\theta. \tag{6.8}$$

The pressure as a function of the depth z is

$$P(z) = g\rho z.$$

The area for $[z, z + dz]$ is

$$dA = \frac{L\,dz}{\cos\theta}.$$

The total force on the wall due to the pressure of the water is

$$\begin{aligned} F &= \int P(z) dA \\ &= \int_0^h g\rho z \frac{L}{\cos\theta} dz \\ &= \frac{1}{2} g\rho h^2 \frac{L}{\cos\theta}. \end{aligned} \tag{6.9}$$

The vertical component is

$$F_z = F \sin\theta = \frac{1}{2} g\rho h^2 L \frac{\sin\theta}{\cos\theta},$$

which is equal to the force of gravity of the water directly above the wall [Eq. (6.8)].

6.2 Archimedes law

"The upward pressure or buoyant force of a submerged object is equal to the weight of the displaced fluid",

$$F_b = gV\rho_{\mathrm{f}}. \tag{6.10}$$

To derive this result, consider a hollow vertical cylinder with length l and area $A = \pi r^2$, with r the radius of the cylinder, submerged in water. If the top of the cylinder is at depth d, then the bottom is at depth $d + l$. By Pascal's law, the pressure at the top is

$$P_1 = g\rho_{\mathrm{f}} d \tag{6.11}$$

and at the bottom

$$P_2 = g\rho_{\mathrm{f}}(d + l) \tag{6.12}$$

The downward force on the cylinder at the top is

$$F_1 = P_1 A \tag{6.13}$$

and the upward force at the bottom is

$$P_2 = P_2 A \tag{6.14}$$

so the total upward force is

$$F = (P_2 - P_1)A = g\rho_{\mathrm{f}} lA = g\rho_{\mathrm{f}} V = gM, \tag{6.15}$$

with M the mass of the water with density ρ in the volume of the cylinder (V).

The combined upward force of buoyancy and gravity for an object with density ρ_{obj} is

$$\sum F = F_b - F_g = gV(\rho_{\mathrm{f}} - \rho_{\mathrm{obj}}). \tag{6.16}$$

For an iceberg, which floats in the water we have

$$F_g = F_b \tag{6.17}$$

$$g\rho_{\mathrm{obj}} V_{\mathrm{obj}} = g\rho_{\mathrm{f}} V_{\mathrm{disp}} \tag{6.18}$$

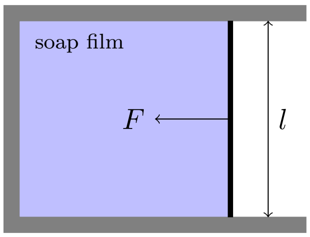

Figure 6.2: A soap film exerting a force due to surfaces tension.

so the fraction of the volume of the iceberg that is submerged is

$$\frac{V_{\text{disp}}}{V_{\text{obj}}} = \frac{\rho_{\text{obj}}}{\rho_{\text{f}}}. \tag{6.19}$$

6.3 Surface tension

A video of a water bubble in the space station shows that it settles into a sphere. For a given volume, a sphere has a smaller area than any other shape. Thus, we can model this effect by assuming that the surface layer of the bubble behaves like an elastic membrane that tries to minimize the surface area.

We assume that the potential energy U is proportional to the surface area A,

$$U = \gamma A, \tag{6.20}$$

where γ is the **surface tension** which has units $J/m^2 = N/m$.

A soap film [see Fig. (6.2)] exerts a force on the rod with length l. Displacement of the rod gives a change of the area by $dA = l\,dx$

$$F = -\frac{\partial U}{\partial z} = \frac{\partial A}{\partial z}\frac{\partial U}{\partial A} = -2\gamma l. \tag{6.21}$$

The factor of two arises because the film has two sides.

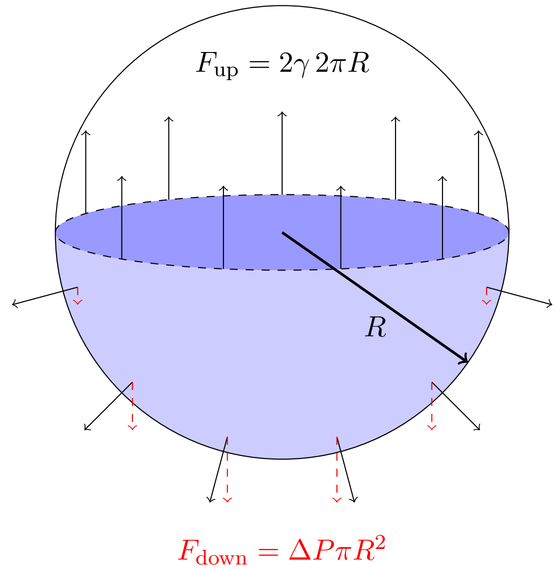

Figure 6.3: Surface tension in soap bubble.

In a soap bubble [Eq. (6.3)] the surface tension contracts the bubble, which makes the pressure inside the bubble ΔP higher than outside. We derive the relation between ΔP and γ in equilibrium in two ways, neglecting gravity in both cases.

The potential energy due to the surface tension for a bubble with radius R is

$$U(R) = 2\gamma\, 4\pi R^2, \tag{6.22}$$

again with a factor of two because the soap film has two sides. The inward force due to surface tension is

$$F_{\text{soap}} = -\frac{\partial U}{\partial R} = 16\gamma\pi R. \tag{6.23}$$

The outward force due to pressure is

$$F_{\text{pressure}} = 4\pi R^2 \Delta P. \tag{6.24}$$

In equilibrium these forces are equal, which gives

$$\Delta P = \frac{4\gamma}{R}. \tag{6.25}$$

This is the **Young-Laplace equation**. Note that sometimes, e.g. on Wikipedia, γ is defined as the **wall tension**, which is equal to our 2γ.

Another way to find this relation is consider the top half and the bottom half of the bubble separately [Eq. (6.3)]. Then, the bottom half of the bubble is pulling on the "equator" due to the pressure on the bottom half of the sphere,

$$F_{\text{down}} = \pi R^2 \Delta P. \tag{6.26}$$

The area that we use in the formula is the area of a circle with radius R. The actual area of the bottom half of the bubble is twice as large. However, we only take into account the vertically downward component of the force. It is left as an exercise to show that, integrated over half the sphere, this gives a factor of $1/2$. In equilibrium, the downward force due to pressure must be equal to the upward on the equator due to surface tension,

$$F_{\text{up}} = 2\gamma 2\pi R. \tag{6.27}$$

Equating F_{down} and F_{up} gives again the relation between ΔP and γ of Eq. (6.25).

6.4 Contact angle

Figure 6.4 shows a droplet on a solid surface. In addition to the interface between the liquid and the gas, with surface tension γ_{SG}, there is an interface between the liquid and the solid, with surface tension γ_{LS}. The energy per area for the interface between the solid and the gas, γ_{SG} is usually not called a surface tension, because a

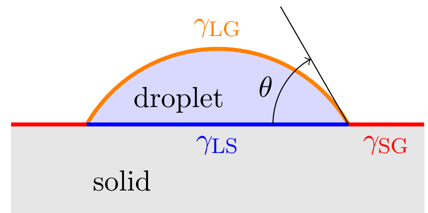

Figure 6.4: Contact angle (θ) of a droplet on a solid surface.

solid does not adapt its shape to minimize the area.

The contact angle θ can be related to the surface tensions and energies with arguments from mechanics and from thermodynamics[2]. In Serway[1], section 13.6, mechanics is used. Also, Serway does not introduce γ_{SG}, but in Eq. (13.11) defines γ_{LS} to be the difference between the liquid-gas surface tension and γ_{SG}.

Increasing the interface area A_{LS} between the solid and the droplet reduces the area between the solid and the gas, so the change in potential energy is

$$dU = (\gamma_{\text{LS}} - \gamma_{\text{SG}})dA. \tag{6.28}$$

We take the area of contact surface between the droplet and the solid to be a disc with radius r, then

$$dA = 2\pi r dr, \tag{6.29}$$

so the inward force due to $\gamma_{\text{LS}} - \gamma_{\text{SG}}$ on the radius of the disc is

$$F_{\text{S}} = -\frac{dU}{dr} = 2\pi r(\gamma_{\text{LS}} - \gamma_{\text{SG}}). \tag{6.30}$$

The additional force due to surface tension γ_{LG} is

$$F_{\text{LG}} = 2\pi r \gamma_{\text{LG}} \cos\theta \tag{6.31}$$

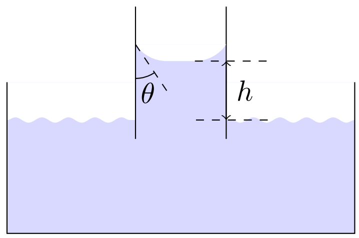

Figure 6.5: Capillary action.

so in equilibrium we have

$$\gamma_{LS} - \gamma_{SG} = -\gamma_{LG} \cos\theta. \tag{6.32}$$

This is the **Young-Dupré equation**.

6.5 Capillary action

Figure 6.5 shows a capillary tube in a cylinder with water. The water level in the tube is raised a distance h above the level in the cylinder. The meniscus in the tube is **concave**, and the contact angle between the water and the surface is θ.

The rising of the water in the capillary tube increases the liquid-solid area and reduces solid-gas area by

$$A_S = 2\pi r h, \tag{6.33}$$

so the surface energy is

$$U_S = A_S(\gamma_{LS} - \gamma_{SG}). \tag{6.34}$$

In the case of water and glass, the **adhesive** interaction between the water and the glass is larger than the **cohesive** interaction between the water molecules, so $U_S < 0$, i.e., the forces try to increase A and hence also h

$$F_S = -\frac{dU_S}{dh} = -2\pi r(\gamma_{LS} - \gamma_{SG}). \tag{6.35}$$

This force is counteracted by the force of gravity, which is

$$F_g = mgh, \tag{6.36}$$

with the mass of the water above the level of the water in the cylinder

$$m = \rho V = \rho \pi r^2 h. \tag{6.37}$$

Equating F_S and F_g gives

$$h = -\frac{2(\gamma_{LS} - \gamma_{SG})}{r\rho g}. \tag{6.38}$$

With Young-Dupré [Eq. (6.32)] this gives

$$h = \frac{2\gamma_{LG}\cos\theta}{r\rho g}, \tag{6.39}$$

as in Serway[1] Eq. (13.12). Note that although in this equation the water-air surface tension γ_{LG} appears, it is the adhesive interaction between the glass surface and the water and the energy of the glass-air surface that lift the water.

Chapter 7

Fluid dynamics

We consider **laminar flow** of a fluid. We assume the fluid to be **non-viscous**, which means there is no friction in the fluid, i.e., the flow does not convert energy into heat. We will also assume the flow is **irrotational**, which means the fluid has no angular momentum. We will also assume the fluid is **non-compressible**, which means that the density (ρ) is constant.

At high velocity, laminar flow turns into **turbulent** flow, but we will not consider that here.

Water is an example of a fluid with low viscosity. Water flowing with velocity v through a pipe with cross section area of A has a **volume rate of flow** or **flux** of

$$Q = vA. \tag{7.1}$$

The unit of flux is m^3/s. If the cross section area A of the pipe increased at some point, the velocity decreases such that the flux is constant. This is expressed by the **continuity** equation

$$A_1 v_1 = A_2 v_2, \tag{7.2}$$

where the subscripts 1 and 2 refer to different cross sections of the pipe.

7.1 Bernoulli equation

(Serway[1], Fig. 14.6) We consider water (or, in general, a fluid), flowing through a section of a pipe. The entrance of the pipe is at height h_1, the cross section has area A_1, the pressure is P_1, and the velocity of the water is v_1. At the exit of the pipe the area is A_2, the height is h_2, the pressure is P_2, and the velocity is v_2. The increase in height is $h = h_2 - h_1$ and the density of the water is ρ.

In a time interval Δt, the volume of water that enters the pipe is

$$V = Q\Delta t = v_1 A_1 \Delta t, \tag{7.3}$$

and, since the density is assumed constant, this is also the volume that exits the pipe in that time interval. Work is done on the water by the pressure at the entrance. The force is $F_1 = P_1 A_1$, the work done is $dW_1 = F_1 \Delta x_1$, and $\Delta x_1 = v_1 \Delta t$, so

$$dW_1 = P_1 A_1 v_1 \Delta t = P_1 V_1 \tag{7.4}$$

and the total work is

$$dW = (P_1 - P_2)V. \tag{7.5}$$

The mass entering the pipe in time interval Δt is

$$M = \rho V. \tag{7.6}$$

The work W done on the water in the pipe increases the sum of kinetic T and potential energy U of the water. The change in kinetic energy in the time interval Δt is

$$T = \frac{1}{2}M(v_2^2 - v_1^2) = \frac{1}{2}\rho V(v_2^2 - v_1^2). \qquad (7.7)$$

The change in potential energy is

$$U = gMh = gh\rho V. \qquad (7.8)$$

Thus,

$$W = T + U \qquad (7.9)$$

divided by the volume V is

$$P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + gh\rho. \qquad (7.10)$$

With $h = h_2 - h_1$ we can also write this as

$$P_1 + \frac{1}{2}\rho v_1^2 + g\rho h_1 = P_2 + \frac{1}{2}\rho v_2^2 + g\rho h_2, \quad (7.11)$$

or

$$P + \frac{1}{2}\rho v^2 + g\rho h = \text{constant}. \qquad (7.12)$$

If the diameter of the pipe is constant, this shows that $P + g\rho h$ is constant, i.e., when the water flows up, the pressure must go down.

At a fixed height, this gives $P + \frac{1}{2}\rho v^2$ is constant, so if the velocity increases, i.e., when the diameter of the pipe decreases, the pressure get **lower**. This effect is used in a **Venturi** tube to measure the flow rate or create a pump. The effect also explains the lift of an airplane wing as the air flows faster above the wing than below, resulting in an upward pressure.

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