Classical mechanics for chemists

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I. INTRODUCTION

These lecture notes of the course "Mechanics, electricity, and Magnetism 2", as given in the first quarter of 2025. It does not replace the text book (Serway), but it summarizes the equations and main ideas.

II. CLASSICAL MECHANICS IN ONE DIMENSION

In classical mechanics a system of n particles is defined by the **masses** of the particles, $\{m_i, i = 1, 2, \ldots, n\}$ and their **positions** as a function on time (t), $\{x_i(t), i = 1, 2, \ldots, n\}$. We will take $x_i(t)$ to be the Cartesian coordinate of particle i with respect to an **inertial** frame, as defined in **Newton's first law**. The function $x_i(t)$ is called the **trajectory** or also the **orbit**.

The time derivative of the position is the velocity

$$v(t) = \dot{x}(t) = \frac{d}{dt}x(t) \tag{1}$$

and the time derivative of the velocity is the acceleration

$$a = \dot{v} = \ddot{x}. \tag{2}$$

III. NEWTON'S LAWS

Newton's first law defines an **inertial frame**. The main idea is that if there are no forces acting on a particle, it should be at rest or move with a constant velocity with respect to that frame.

Newton's second law relates the acceleration to the forces acting on the particle

$$F = ma. \tag{3}$$

If several forces F_i are acting on a particle than the acceleration is is related to the sum of forces

$$\sum F = ma. \tag{4}$$

Newton's third law concerns the interaction of two particles. If particle 1 excerts of force F on particle 2, then particle 1 experiences a force $-F$ as a result of the interaction.

IV. CALCULATING A TRAJECTORY

With Newton's laws we can learn about forces by observing trajectories. A particle with initial velocity $v(t_i) = v_i$ experiencing a constant force F. From Newton's second law we get the time derivative of the velocity

$$\dot{v}(t) \equiv a(t) = \frac{1}{m}F(t), \tag{5}$$

and by integration we find

$$v(t) = v_i + \frac{1}{m}\int_{t_i}^{t} F(t)dt. \tag{6}$$

Once we know the velocity at every time t, we can compute the trajectory if we know the position $x(t_i) = r_i$ by integrating again

$$x(t_f) = x_i + \int_{t_i}^{t_f} v(t)dt. \tag{7}$$

When the force is actually contant, $F(t) = F_g$, we find that the velocity changes linearly in time

$$v(t) = v_i + \frac{t - t_i}{m}F_g. \tag{8}$$

An example would be the trajectory of particle dropping under the force of **gravity**. Using z rather than x for vertical motion we find that if initially $z(t_i) = z_i$ the trajectory is

$$z(t) = z_i + v_i(t_f - t_i) + \frac{(t - t_i)^2}{2m}F_g. \tag{9}$$

From experiment we find that the trajectory is independent of the mass and the force is pointing down, so we conclude that

$$F_g = -mg, \tag{10}$$

where $g \approx 9.8$ m/s^2, so that the accelaration does not depend on the mass, and matches observation

$$a = \ddot{z} = -g. \tag{11}$$

If the force is constant or if it depends on time we can find the trajectory by integration. It is also possible that the force depends on the position, $F_h = F_h(z)$. For a mass attached to a spring we can have, e.g.,

$$F_h(z) = -k(z - z_e), \tag{12}$$

where k is the **force constant**, and z_e is the **equilibrium position** where the force exerted by the spring is zero. By Newton's second law we know that the **sum of forces** determines the trajectory, so if we already know the force of gravity, we can determine the force constant

k by finding the value of $z = z_s$ for which the mass is stationary

$$F(z_s) + F_g = 0 \tag{13}$$
$$-k(z_s - z_e) - mg = 0 \tag{14}$$

so

$$k = \frac{z_e - z_s}{mg}. \tag{15}$$

Note that by Newton's second law, the trajectory depends on the **sum of all forces** acting on the particle. This is very powerfull: every time a trajectory does not match what we expect based on the forces we already know, we learn about a new force.

Now that we know the force as a function of position, we can try to predict the trajectory if we pull the mass to an initial position z_i and give it an initial velocity v_i. Define the sum of forces

$$F(z) = F_h(z) + F_g \tag{16}$$

we need to solve

$$F(z) = m\ddot{z}. \tag{17}$$

Integration over time as we did before no longer works, since we would need to $z(t)$ to find $F(t) = F(z(t))$. Instead, we now have a second order differential equation with two initial conditions. In our example of a spring there is a simple analytic solution, but in general numeric methods are needed. Sometimes it helps to turn the second order differential equation into two **coupled** first order differential equations

$$\dot{z}(t) = v(t) \tag{18}$$
$$\dot{v}(t) = \frac{1}{m}F(z). \tag{19}$$

In this lecture our aim is not to solve differential equations, but rather to try to simplify the problem as much as possible by using **conservation laws**. Usually, this will not give us the exact trajectory, but we may answer a simpler question, like what is the highest and lowest point of the mass on the spring, or later, what is the velocity of particles after a collision if we know their velocities before the collision.

V. ENERGY CONSERVATION

We can derive conservation laws by integrating Newton's second law. First, let us assume that we have a force $F(z)$ that depends on **position** only. Integrating Newton's second law then gives

$$\int_{z_i}^{z_f} F(z)dz = \int_{z_i}^{z_f} ma\,dz. \tag{20}$$

To integrate the left-hand-side (lhs) we define $-V(z)$ as the primitive of the force, i.e.,

$$F(z) = -\frac{d}{dz}V(z). \tag{21}$$

With these assumptions the integral is

$$\int_{z_i}^{z_f} F(z)dz = -\int_{z_i}^{z_f} \frac{d}{dz}V(z)\,dz = V(z_i) - V(z_f). \tag{22}$$

Not every force be we written as the derivative of a function that only depends on the position, but if it does, we call the force **conservative**. The function $V(z)$ is called **potential energy**. In the example of gravity we find potential energy

$$V_g(z) = mgz, \tag{23}$$

which says that the potential energy is larger the higher you go, and the bigger the mass, which may sound reasonable. The point of using the word **energy** though is that the total energy is **conserved** and that you may **convert** one form of energy into another.

Next, we solve the right-hand-side (rsh) of Eq. (20) Here comes a little **trick**: we can change an integral over **position** into an integral **time** using

$$\frac{dz}{dt} = v(t) \tag{24}$$

so

$$dz = vdt \tag{25}$$

At the same time, we have to change the limits of the integral from initial and final **position** to initial and final **time**, so

$$\int_{z_i}^{z_f} ma\,dz = m\int_{t_i}^{t_f} \dot{v}vdt \tag{26}$$

Next, we use

$$\frac{d}{dt}v^2 = 2v\dot{v}, \tag{27}$$

so the rhs of Eq. (26) becomes

$$\frac{1}{2}m\int_{t_i}^{t_f}\left(\frac{d}{dt}v^2\right)dt = \frac{1}{2}m(v_f^2 - v_i^2). \tag{28}$$

Substituting this result into the rhs of Eq. (20), expressing the lhs in terms of potential energy and rearranging the equation such that everything that depends on the initial state is on the left and everything that depends on the final state is on the right gives

$$\frac{1}{2}mv_i^2 + V(z_i) = \frac{1}{2}mv_f^2 + V(z_f). \tag{29}$$

This equation motivates the definition of **kinetic energy**

$$T = \frac{1}{2}mv^2, \tag{30}$$

so that for **conservative forces** we have a total energy

$$E = T + V(z), \tag{31}$$

which is conserved.

The integral over the force in Eq. (20) is called the **work** done on the particle. The definition of work is more general though, since it also applies to non-conservative forces, which we will discuss later.

VI. LINEAR MOMENTUM

Integrating Newton's second law over position gave us the energy conservation law for converting between potential and kinetic energy. Now, let's see what we get from integrating Newton's second law over time

$$\int_{t_i}^{t_f} F(t)dt = \int_{t_i}^{t_f} ma\, dt. \tag{32}$$

First, let's do the rhs. Using $a = \dot{v}$ gives

$$\int_{t_i}^{t_f} ma\, dt = m \int_{t_i}^{t_f} \dot{v} dt = m(v_f - v_i). \tag{33}$$

We do not yet have a conservation law, but this equation motivates the definition of **linear momentum**

$$p \equiv mv, \tag{34}$$

so the rhs is the **change in linear momentum** between the initial and final time. It may seem there is not much we can do on the lhs of Eq. (32), so we can give it a name: **impulse**,

$$I \equiv \int_{t_i}^{t_f} F(t)dt = p_f - p_i. \tag{35}$$

The good thing about this quantity is that it is defined for an arbitrary force $F(t)$, i.e., it is not restricted to conservative forces. As long as the particle has a trajectory, $z(t)$, it has some acceleration at every time t, and we find the force from Newton's second law. Thus, we may also, write

$$I \equiv \int_{t_i}^{t_f} F\, dt, \tag{36}$$

where now the force may be know directly as $F(t)$, or implicitly as $F[z(t)]$. This is the definition given in Serway.

We also see that if the force is zero the impulse is zero, and linear momentum is conserved. However, we already knew from Newton's second law that if there is no force the velocity is constant, and so the linear momentum is constant. Still, we now find that if the force itself is not zero, but the integral over time, i.e. the impulse is zero, the linear momentum is conserved. Furthermore, we can define an **average force**

$$\overline{F} = \frac{\int_{t_i}^{t_f} F(t)dt}{t_f - t_i}. \tag{37}$$

Then by giving the average force $\overline{F}$ and the time interval $t_f - t_i$ we have specified the impulse

$$I = (t_f - t_i)\overline{F}. \tag{38}$$

Note that here by average, we mean **average over time**.

Many problems are easier when using momenta, rather than velocities, so we already rewrite Newton's second equation in momenta.

$$\sum F = \dot{p}. \tag{39}$$

The concepts of impulse and momentum turn into a real conservation law if we consider the collisions of **two particles**

VII. CONSERVATION OF LINEAR MOMENTUM

Let's consider two particles, labeled 1 and 2, which move along the same straight line with trajectories $x_1(t)$ and $x_2(t)$ and which have masses m_1 and m_2, respectively. From Newton's third law, we know that

$$F_{12} = -F_{21}, \tag{40}$$

where F_{12} is the force exerted by particle 1 on particle 2, and F_{21} is the force exerted by particle 2 on particle 1. This immediately tells us that the impulses on the particles due to their interaction are opposite in sign, so with we get for the change in linear momenta during the collision

$$p_{1,f} - p_{1,i} = -(p_{2,f} - p_{2,i}). \tag{41}$$

We can rewrite this as the law of **conservation of linear momentum**

$$P \equiv p_{1,i} + p_{2,i} = p_{1,f} + p_{2,f}, \tag{42}$$

so the total linear momentum before the collison, P, is equal to the total linear momentum after the collision.

We can also express this relation using velocities

$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}. \tag{43}$$

VIII. ELASTIC COLLISIONS IN 1D

Assuming the velocities of **two particles** before the collision are known, we try to find the velocities of the two particles after the collision. Having two unknowns, we can solve the problem if we have two independent equations. We can certainly use conservation of linear momentum [Eq. (42)], since it was derived from the impulse [Eq. (36)], where we did not make any assumption about the origin of the force. Instead of coordinates $z_1(t)$ and $z_2(t)$ we will use $x_1(t)$ and $x_2(t)$, because we are not thinking about gravity here, but of course this choice is arbitray.

To continue, we assume that the forces between the particles are **conservative**. This means that we can compute the forces from a potential. For two particles a potential energy is function of both coordinates, so we have $V_{12}(x_1, x_2)$, and the forces are given by

$$F_1(x_1, x_2) = -\frac{\partial}{\partial x_1} V_{12}(x_1, x_2) \quad (44)$$

$$F_2(x_1, x_2) = -\frac{\partial}{\partial x_2} V_{12}(x_1, x_2) \quad (45)$$

If the force is the result of, e.g., a spring, the potential will only depend of the distance between the particles, $x = x_2 - x_1$,

$$V_{12}(x_1, x_2) = V(x_2 - x_1) \quad (46)$$

Then we find

$$F_1(x_1, x_2) = -\frac{\partial}{\partial x_1} V(x) = -\frac{dx}{dx_1}\frac{d}{dx} V(x) = V'(x) \quad (47)$$

and

$$F_2(x_1, x_2) = -\frac{\partial}{\partial x_2} V(x) = -V'(x) = -F_1(x_1, x_2). \quad (48)$$

Note that we just recovered Newton's third law for the special case of conservative forces! We now assume that the potential is zero when the particles are sufficiently far apart. That means we can take time t_i sufficiently long before the collision, so the energy of the system is just the sum of the kinetic energies of the particles. We take time t_f when the particles are moving apart and the potential energy $V(x_2 - x_1)$ has again dropped to zero, so we have conservation of kinetic energy

$$\frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} m_2 v_{2,i}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2. \quad (49)$$

Strictly, we only derived above that the sum of kinetic and potential energy for a **single particle** is conserved. We will come back to this later and solve the **elastic collision** problem by solving the two equations with the two unknowns.

Since conservation of linear momentum is the simplest when written in terms of linear momenta, we rewrite the conservation of kinetic energy (for elastic collisions) also using momenta. In the final step of the calculation we will convert the momenta back to velocities. For the kinetic energy of a single particle we have

$$T = \frac{1}{2} m v^2 = \frac{p^2}{2m} \quad (50)$$

so for two particles we have

$$E = T_1 + T_2 = \frac{p_{1,i}^2}{2m_1} + \frac{p_{2,i}^2}{2m_2} = \frac{p_{1,f}^2}{2m_1} + \frac{p_{2,f}^2}{2m_2}. \quad (51)$$

We can turn this equation with two unknowns, $p_{1,f}$ and $p_{2,f}$, into a single equation with one unknown by substituting $p_{2,f} = P - p_{1,f}$ when we find the total linear momenta from $P = p_{1,i} + p_{2,i}$. Thus, we need to solve an equation quadratic in $p_{1,f}$,

$$E = \frac{p_{1,f}^2}{2m_1} + \frac{(P - p_{1,f})^2}{2m_2}. \quad (52)$$

To work out the general expression is quite a bit of work, although when you plug in the numbers this approach may be ok. The book (Serway) solves the problem using velocities rather than momenta in the derivation, but it shows a little trick make deriving the general expression easier, that we can use too. We rewrite the energy conservation law as

$$\frac{p_{1,f}^2 - p_{1,i}^2}{2m_1} = -\frac{p_{2,f}^2 - p_{2,i}^2}{2m_2} \quad (53)$$

so we have the change in kinetic energy of the particle on the left, and minus the change in kinetic energy of particle 2 on the right. Next, we factorize the difference of squares

$$\frac{(p_{1,f} + p_{1,i})(p_{1,f} - p_{1,i})}{2m_1} = -\frac{(p_{2,f} + p_{2,i})(p_{2,f} - p_{2,i})}{2m_2}. \quad (54)$$

We can now use the conservation of linear momentum [Eq. (41)] to divide the lhs by the change in linear momentum of particle 1 and the rhs by minus the change in the linear momentum of particle two. This way we get two linear equations with two unknowns, which are just a bit easier to solve than a single linear equation with one unknown

$$p_{1,f} - p_{1,i} = -(p_{2,f} - p_{2,i}) \quad (55)$$

$$\frac{p_{1,f} + p_{1,i}}{2m_1} = \frac{p_{2,f} + p_{2,i}}{2m_2} \quad (56)$$

Multiplying the second equation with $2m_2$ and adding the equations gives a linear equation in $p_{1,f}$

$$(\frac{m_2}{m_1} + 1)p_{1,f} + (\frac{m_2}{m_1} - 1)p_{1,i} = 2p_{2,i} \quad (57)$$

Multiplying with m_1 and rearranging gives

$$p_{1,f} = \frac{(m_1 - m_2)p_{1,i} + 2m_1 p_{2,i}}{M}, \quad (58)$$

where $M = m_1 + m_2$ is the **total mass**. Dividing by m_1 we can rewrite this in velocities

$$v_{1,f} = \frac{(m_2 - m_1)v_{1,i} + 2m_2 v_{2,i}}{M}. \quad (59)$$

By multiplying Eq. (56) with m_1 and adding the two equations, we can derive in a similar way the expression for $p_{2,f}$. Alternatively, we can use conservation of linear momentum

$$p_{2,f} = \underbrace{p_{1,i} + p_{2,i}}_{\text{total momentum}} - p_{1,f} \quad (60)$$

so

$$p_{2,f} = \frac{(m_1+m_2)(p_{1,i}+p_{2,i})+(m_2-m_1)p_{1,i}-2m_1p_{2,i}}{M}$$
$$= \frac{2m_2p_{1,i}+(m_2-m_1)p_{2,i}}{M} \quad (61)$$

Dividing by m_1 and rewriting in velocities gives

$$v_{2,f} = \frac{2m_1v_{1,i}+(m_2-m_1)v_{2,i}}{M}. \quad (62)$$

For the special case of $m_1 = m_2$ this simplifies to

$$p_{1,f} = p_{2,i} \quad (63)$$
$$p_{2,f} = p_{1,i} \quad (64)$$

so the momenta swap. Since the masses are equal we also get $v_{1,f} = v_{2,i}$ and $v_{2,f} = v_{1,i}$.

If initially particle 2 is at rest, $p_{2,i} = 0$, we find

$$p_{1,f} = \frac{m_1-m_2}{M}p_{1,i} \quad (65)$$
$$p_{2,f} = \frac{2m_2}{M}p_{1,i}. \quad (66)$$

and also

$$v_{1,f} = \frac{m_1-m_2}{M}v_{1,i} \quad (67)$$
$$v_{2,f} = \frac{2m_2}{M}v_{1,i}. \quad (68)$$

If, in addition, $m_1 \gg m_2$,

$$v_{1,f} = v_{1,i} \quad (69)$$
$$v_{2,f} = 2v_{1,i}. \quad (70)$$

IX. ENERGY CONSERVATION FOR NON-CONSERVATIVE FORCES

We may repeat the derivation of section V, but for the lhs of Eq. (20), the **work**

$$W = \int_{z_i}^{z_f} F dz \quad (71)$$

we do not assume the force is related to a potential energy. The rhs of Eq. (20) still gives the same, so we find that the work gives the difference in kinetic energy

$$W = T_f - T_i. \quad (72)$$

An example of a non-conservative force is **friction**. A particle (or object) sliding over a surface until it comes to rest has lost all its kinetic energy. We could have made a model of the surface consisting of particles (atoms or molecules), interacting through a potential function that depends on all the coordinates. Then, if would take into account the kinetic energy of the atoms in the surface, mechanical energy would be conserved. If we only want to include the sliding particle in the model, we call all the energy that has gone into the surface **heat** and **total energy** defined as **kinetic energy** of the sliding particle **plus** the heat is again conserved. Thus, the term **heat** appears when we want to sweep a lot of detail under the rug. This approach is elaborated in **thermodynamics**, where the concepts of **work** and **heat** are central, but the microscopic detail of trajectories of particles is left out. In **statistical mechanics** a connection is made between **thermodynamics** and a microscopic description of the system. For now, we say that the work done by non-conservative forces, e.g., friction, on the (sliding) particle, is converted to heat.

X. INELASTIC COLLISIONS

In an **inelastic collision**, part of the kinetic energy is lost. In a **perfectly inelastic collision**, the two particles stick together.

When an atom collides with a molecule it may happen that the molecules get rotationally or vibrationally excited. By the laws of quantum mechanics this will happen in discrete energy **quanta**. This energy is no longer available for the translational kinetic energy of the atom and molecule after the collision, so we would call such a collions **inelastic**. Since the energy in the molecule after the collision may not be distributed among all possible vibrations and rotations according to thermodynamics we would not call this energy **heat**. However, if the molecule is large the energy may redistribute eventually, and we may say that the collision has heated the molecule.

In a **perfectly inelastic** collision, where the molecules stick together, we have only one unknown the velocity of the particles after the collision. Since the **law of conservation of linear momentum** applies to both conservative and nonconservative forces we can use it to solve this problem:

$$p_{1,i} + p_{2,i} = p_f \quad (73)$$

where

$$p_f = (m_1 + m_2)v_f \quad (74)$$

The solution is

$$v_f = \frac{m_1v_{1,i} + m_2v_{2,i}}{m_1 + m_2}. \quad (75)$$

XI. CENTER OF MASS

The center of mass (C.O.M.) is the mass weighted average of the position of the particles

$$X(t) = \frac{m_1x_1(t) + m_2x_2(t)}{M}, \quad (76)$$

where again $M = m_1 + m_2$. The velocity of the C.O.M. is the time derivative

$$V(t) = \dot{X}(t) = \frac{m_1 v_1(t) + m_2 v_2(t)}{M} = \frac{p_1 + p_2}{M} = \frac{P}{M} \tag{77}$$

So, for the total momentum we have

$$P = MV. \tag{78}$$

Also, the equation of motion of the C.O.M. is the same as for a particle of mass M, and we find

$$\dot{P} = \dot{p}_1 + \dot{p}_2 = \sum F_1 + \sum F_2 = \sum F, \tag{79}$$

where the sum over forces includes all forces acting on either particle 1, or particle 2. The forces that arise from the interactions between the particles cancel by Newton's third law, so in fact, we may sum over external forces only

$$\dot{P} = \sum F_{\text{ext}}, \tag{80}$$

and if there are no external forces, as in the collision problems we are studying, we find the P is constant, as before.

Note that the C.O.M. velocity V is equal to $v_{1,f} = v_{2,f}$ in a **perfectly inelastic** collision.

XII. CENTER OF MASS FRAME

The equation of motion of the C.O.M. is relatively simple, since it does not depend on internal forces. Therefore, it is worthwhile to switch for coordinates x_1 and x_2 to a new set of coordinates, the C.O.M. coordinates

$$X = \frac{m_1 x_1 + m_2 x_2}{M} \tag{81}$$

and a relative coordinate

$$x = x_2 - x_1. \tag{82}$$

We can solve these equations for x_1 and x_2,

$$x_1 = X - \frac{m_2 x}{M} \tag{83}$$

$$x_2 = X + \frac{m_1 x}{M}. \tag{84}$$

This result is easily verified by substituting it back into Eqs. (81) and (82).

We already have an equation of motion for the C.O.M. X. For the distance between the particles we have

$$\ddot{x} = \ddot{x}_2 - \ddot{x}_1 \tag{85}$$

$$= \frac{F_2}{m_2} - \frac{F_1}{m_1}. \tag{86}$$

Now consider **conservative forces** so we have

$$F_1 = -F_2 = V'(x) \equiv F \tag{87}$$

where we define the internal force F. Then

$$\ddot{x} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) F = \frac{m_1 + m_2}{m_1 m_2} F. \tag{88}$$

Thus, by defining the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{89}$$

(does μ have the dimension mass?), we find Newton's equation of motion for $x(t)$,

$$F = \mu \ddot{x}. \tag{90}$$

Thus, equations of motion for $X(t)$ and $x(t)$ are not coupled, with $X(t)$ depending on external forces, and $x(t)$ depending on internal forces. The problem of solving $x(t)$ is identical to solving the motion of a particle with mass μ driven by potential energy $V(x)$. Hence, we can define a velocity $v = \dot{x}$ and momentum p is

$$p = \mu v. \tag{91}$$

After solving the internal motion $x(t)$, $v(t)$, and $p(t)$ we can find the momenta of the two particles

$$p_1 = m_1 \dot{x}_1 = m_1 \dot{X} - \frac{m_1 m_2}{M}\frac{p}{\mu} = \frac{m_1}{M} P - p$$

$$p_2 = m_2 \dot{x}_2 = m_2 \dot{X} + \frac{m_2 m_1}{M}\frac{p}{\mu} = \frac{m_2}{M} P + p \tag{92}$$

and for the velocities

$$v_1 = \frac{p_1}{m_1} = V - \frac{p}{m_1} \tag{93}$$

$$v_2 = \frac{p_2}{m_2} = V + \frac{p}{m_2}. \tag{94}$$

The kinetic energy, expressed in C.O.M. momenta is

$$T = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{p^2}{2\mu}. \tag{95}$$

So the kinetic energy is the sum of a contribution from the C.O.M. and a contribution from the relative motion.

XIII. ELASTIC COLLISION IN 1D, USING C.O.M. FRAME

We assume that the initial velocities, $v_{1,i}$ and $v_{2,i}$ are known. For the C.O.M. velocity we find with Eq. (77)

$$V = \frac{m_1}{M} v_{1,i} + \frac{m_2}{M} v_{2,i}. \tag{96}$$

The initial relative velocity is

$$v_i = v_{2,i} - v_{1,i}. \tag{97}$$

The initial relative momemtum is

$$p_i = \mu v_i = \frac{m_1 m_2}{m_1 + m_2} v_i. \tag{98}$$

Due to the collision, **the relative velocity and momentum change sign**:

$$p_f = -p_i. \tag{99}$$

With Eqs. (93) and (94) we find the final velocities in the lab frame

$$v_{1,f} = V - \frac{p_f}{m_1} \tag{100}$$

$$v_{2,f} = V + \frac{p_f}{m_2}. \tag{101}$$

Exercise: check that these results match Eqs. (59) and (62)

Note: in quantum mechanics the equations of motion are very different. Still, the C.O.M. motion and the relative motion can be solved seperately.

XIV. MOTION IN 2D

Much of what we did so far can readily be extended to 2D. We must keep in mind though, that a trajectory now has two components

$$\boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{102}$$

and the velocity $\boldsymbol{v}(t)$, the acceleration $\boldsymbol{a}(t)$, linear momentum $\boldsymbol{p}(t)$, the force $\boldsymbol{F}(t)$, and the impulse $\boldsymbol{I}$, now all become vectors with an x and a y component. Masses and energies remain scalars. So Newton's second law in 2D is

$$\boldsymbol{F} = m\boldsymbol{a} = \dot{\boldsymbol{p}}. \tag{103}$$

For a **conservative force** the derivative of the potential becomes a gradient

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r}) = -\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} V(\boldsymbol{r}). \tag{104}$$

Integrating Newton's second law over time now requires integrals over x and y, and the results are similar, e.g., conservation of linear momentum turns into conservation of linear momentum in the x- and y- direction.

To derive the relation between work and kinetic energy we need

$$\int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} m\boldsymbol{a} \cdot d\boldsymbol{r} \tag{105}$$

and to turn the integral over position into an integral over we need use

$$d\boldsymbol{r} = \boldsymbol{v}\, dt \tag{106}$$

and

$$\boldsymbol{a} \cdot \boldsymbol{v} = \frac{1}{2}\frac{d}{dt}\boldsymbol{v} \cdot \boldsymbol{v}. \tag{107}$$

For the integral over the force we use

$$\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{v}\, dt \tag{108}$$

This can be rewritten as a **total derivative** with respect to t,

$$\frac{d}{dt}V(\boldsymbol{r}) = \left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}\right) V(\boldsymbol{r}) \tag{109}$$

$$= \boldsymbol{v} \cdot \boldsymbol{\nabla} V(\boldsymbol{r}) = -\boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{v} \tag{110}$$

so

$$\int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{t_i}^{t_f} \frac{d}{dt}V(\boldsymbol{r})dt = V(\boldsymbol{r}_i) - V(\boldsymbol{r}_f). \tag{111}$$

The same method can be used to prove the **conservation of mechanical energy for two particles** moving on a straight line interacting with a **conservative force**, as in Eq. (49).

XV. C.O.M. FRAME IN 2D

The C.O.M. and relative coordinates are

$$\boldsymbol{X} = \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2 \tag{112}$$

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1. \tag{113}$$

Transforming back to the lab frame

$$\boldsymbol{r}_1 = \boldsymbol{X} - \frac{m_2}{M}\boldsymbol{r} \tag{114}$$

$$\boldsymbol{r}_2 = \boldsymbol{X} + \frac{m_1}{M}\boldsymbol{r}. \tag{115}$$

For the relative momentum we have

$$\boldsymbol{p} = \mu\boldsymbol{v} = \mu\dot{\boldsymbol{r}} \tag{116}$$

and the momentum associated with the C.O.M. motion is the total linear momentum

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = M\boldsymbol{V}. \tag{117}$$

After solving $\boldsymbol{r}(t)$, $\boldsymbol{v}(t)$, and $\boldsymbol{p}(t)$ we can tranform the result back to the lab frame, as in Eqs. (92)

$$\boldsymbol{p}_1 = m_1\dot{\boldsymbol{r}}_1 = \frac{m_1}{M}\boldsymbol{P} - \boldsymbol{p} \tag{118}$$

$$\boldsymbol{p}_2 = m_2\dot{\boldsymbol{r}}_2 = \frac{m_2}{M}\boldsymbol{P} + \boldsymbol{p} \tag{119}$$

and for the velocities

$$\boldsymbol{v}_1 = \frac{\boldsymbol{p}_1}{m_1} = \boldsymbol{V} - \frac{\boldsymbol{p}}{m_1} \tag{120}$$

$$\boldsymbol{v}_2 = \frac{\boldsymbol{p}_2}{m_2} = \boldsymbol{V} + \frac{\boldsymbol{p}}{m_2}. \tag{121}$$

The total kinetic energy is, as before

$$T = \frac{P^2}{2M} + \frac{p^2}{2\mu}, \tag{122}$$

except that now, $P^2 = \boldsymbol{P} \cdot \boldsymbol{P}$ and $p^2 = \boldsymbol{p} \cdot \boldsymbol{p}$.

XVI. PERFECTLY INELASTIC, 2D, C.O.M. FRAME

After the collision the stick to each other, so

$$\boldsymbol{v}_{1,f} = \boldsymbol{v}_{2,f} = \boldsymbol{V} = \frac{m_1 \boldsymbol{v}_{1,i} + m_2 \boldsymbol{v}_{2,i}}{m_1 + m_2}. \tag{123}$$

XVII. ELASTIC, 2D. C.O.M. FRAME

We have four unknowns, the x and y components of $\boldsymbol{v}_{1,f}$ and $\boldsymbol{v}_{2,f}$. However, we only have three conservation laws: linear momentum in the x-direction, linear momentum in the y-direction, and mechanical energy conservation.

The C.O.M. linear momentum $\boldsymbol{P}$ takes care of the linear momentum conservation. Conservation of kinetic energy [Eq. (122)] gives

$$\frac{p_i^2}{2\mu} = \frac{p_f^2}{2\mu}, \tag{124}$$

since the C.O.M. contributions cancel, so this gives as the length of the vector $|\boldsymbol{p}_f| = p_f$. If, in an experiment we measure the directions after the collision, we can solve for the momenta and velocities of both particles.

$$\boldsymbol{p}_f = p_f \hat{\boldsymbol{e}}_f, \tag{125}$$

where $\hat{\boldsymbol{e}}_f$ is a unit vector. We may specify it with angle ϕ_f as

$$\boldsymbol{e}_f = \begin{pmatrix} \cos\phi_f \\ \sin\phi_f \end{pmatrix}. \tag{126}$$

With the equations in section XV we can than find the velocities $\boldsymbol{v}_{1,f}$ and $\boldsymbol{v}_{2,f}$.