Comment on "Why the speed of light is reduced in a transparent medium," by Mary B. James and David J. Griffiths [Am. J. Phys. 60, 309–313 (1992)]

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In a recent article Mary B. James and David J. Griffiths treated the transmission and reflection of a plane electromagnetic wave normally incident on a transparent medium using a perturbative approach. In this approach, the incident wave electric field (zeroth order) polarizes the medium; the oscillating dipoles associated with this polarization radiate—giving rise to an additional electric field (first order), which in turn produces an additional polarization, etc. The total field in the medium is obtained as an infinite series in powers of the electric susceptibility \( \chi_e \), and when summed, agrees with the usual result derived from the wave equation with appropriate boundary conditions. In this note, we point out a simple interpretation of the first-order perturbation results which may be useful in presenting these ideas to students.

Previous discussions of this type in textbooks have treated light which has passed through a thin slab of dielectric medium. Using the solution of Ref. 1, we consider here the transmitted light traveling within the medium.

Our analysis focuses on the first-order electric field, which includes a term that can be represented as a plane wave with phase lagging 90 deg behind the phase of the incident wave and with amplitude increasing as the wave moves deeper into the medium. When this first-order contribution is added to the incident wave, the resulting field has a phase which lags increasingly behind the incident wave as the wave moves through the medium. This steadily growing phase lag implies a reduction in phase velocity, which is shown to agree to first order in \( \chi_e \) with the usual result for the speed of light in a medium.

In Ref. 1, a plane wave \( E^0 = E_0 e^{i(kx - \omega t)} \) is normally incident from vacuum \((x<0)\) on a linear dielectric medium \((x>0)\). The calculation of the electric field in the medium to first order in \( \chi_e \) may be described as follows [see Eqs. (20) to (24) of Ref. 1]. \( E^1 \) produces a polarization \( P^1 = \varepsilon_0 \chi_e E^1 \) in the medium, which implies that within each plane defined by \( x' = \text{const} > 0 \) the resulting dipoles oscillate identically, in phase with one another. Each such plane of dipoles produces an electric field at the point of observation \( x \), and the total first-order field \( E^1 \) is just the superposition of these fields. The superposition (an integration over \( x' \)) leads to the result:

\[
E^1 = E^0 \left[ \chi_e i k x / 2 + (-\chi_e / 4) E_0 e^{i(kx - \omega t)} \right].
\]

The first term of Eq. (1) arises from the region of polarized medium with \( x' \) such that \( 0 < x' < x \). This region lies between the plane interface \((x'=0)\) and a parallel plane passing through the point \( x \); thus the region has thickness equal to \( x \).

We note two features of this term: it has magnitude proportional to \( x \), and its phase lags that of \( E^0 \) by 90 deg. These features directly affect the phase velocity of the light, and their origin is explored in the next paragraph. The second term of Eq. (1) arises from light reflected back to \( x \) from the remainder of the medium, for which \( x' > x \).

Our goal is to calculate the phase velocity to first order in \( \chi_e \), but before doing so we shall look more closely at the calculation of \( E^1 \) outlined above. Consider a planar slab of medium defined by \( x' = \text{constant} \), with thickness \( dx' \). The oscillating dipoles in this slab produce a time-dependent surface current density \( K \) which is spatially uniform over the slab, given by

\[
K = J_0 dx' = (\partial P^1 / \partial t) dx' = \varepsilon_0 \chi_e (\partial E^0 / \partial t) dx',
\]

where \( J_0 \) is the polarization current density. The contribution \( dE^1 \) to the electric field at position \( x \) and time \( t \) from this slab of polarized medium is

\[
dE^1 = -(\mu_0 c / 2) K (t - |x-x'| / c).
\]

Combining Eqs. (2) and (3) and using the plane-wave expression for \( E^0 \), we get, for \( x' < x \),

\[
dE^1 = -(\mu_0 c / 2) \varepsilon_0 \chi_e (\partial E^0 / \partial t |_{x',t'}) dx' = (ik \chi_e / 2) \varepsilon_0 \chi_e \partial E^0 / \partial t |_{x',t'} dx'.
\]

In the middle expression of Eq. (4), \( \partial E^0 / \partial t \) is evaluated at \( x' \) and \( t' \), where \( t' \) is the retarded time \( t - (x-x') / c \). We see that the contribution \( dE^1 \) from the slab at \( x' \) is a plane wave propagating in the medium at speed \( c \), lagging \( E^0 \) in phase by 90 deg—due to the time derivative of \( E^0 \) involved in obtaining the polarization current density. Since the expression for \( dE^1 \) in Eq. (4) is independent of \( x' \), the contributions from all slabs with \( 0 < x' < x \) are in phase and in fact, identical. Integration over \( x' \) from 0 to \( x \) yields the linear dependence on \( x \) of the first term of \( E^1 \) in Eq. (1). The physical picture that emerges here is that the incident wave \( E^0 \) polarizes a slab at \( x' \), producing a wave \( dE^1 \) which propagates at speed \( c \) along with \( E^1 \), but lagging 90 deg in phase. At the point of observation \( x \), the wave contributions from all the slabs \((x' < x)\) arrive simultaneously (together with \( E^0 \) in phase with each other and 90 deg behind \( E^0 \)—to make up the total field.

We now turn to the calculation of the speed of light in the medium. It is assumed in what follows that both \( \chi_e \) and \( (\varepsilon_0 k x) \) are \(<1\). We write the total electric field \( E \) to first order in \( \chi_e \) as

\[
E = E^0 + E^1 = E_0 + i E_0 (\varepsilon_0 k x / 2) e^{i(\omega - \omega_0 t)}.
\]

where \( E_0 = (\varepsilon_0 k x / 2) E_0 \). The second term of Eq. (1) has been dropped from Eq. (5). This term is in phase with \( E^0 \) and (under our assumptions) negligible in magnitude compared to \( E_0 \). We shall see (Ref. 6) that the dropped term makes only a second-order contribution to the speed of light in the medium.

The sum implied in Eq. (5) can be represented in a phasor diagram (see Fig. 1). Due to the \( e^{-i\omega t} \) phase factor, the phasors shown in the figure rotate clockwise with the first-order field lagging the zeroth-order contribution by \( 90^\circ \). An important quantity is the phase lag \( \alpha \) of \( E \) relative to \( E^0 \). From the diagram, we see that \( \alpha \) is given in radians by

\[
\alpha = E_1 / E_0 = (\varepsilon_0 k x / 2) E_0 / E_0 = \varepsilon_0 k x / 2.
\]
Note that $\alpha$ increases linearly with $x$. The phase $\phi$ of $E$ is then

$$\phi = kx - (\omega t - \alpha) = kx (1 + \chi_e/2) - \omega t = k_1 x - \omega t,$$

(7)

where $k_1 = k(1 + \chi_e/2)$. Finally, the phase velocity $v$ for light in the medium can be obtained (within our approximations) as

$$v = \omega / k_1 = \omega / [k(1 + \chi_e/2)] = c / (1 + \chi_e/2).$$

(8)

The exact result 7 for the speed of light in the medium is $c / (1 + \chi_e)^{1/2}$, which agrees with the last expression in Eq. (8) to first order in $\chi_e$. This outcome is to be expected, given the first-order accuracy of our calculations.

Thus we see that superposing waves (which propagate at speed $c$) due to planes of oscillating dipoles yields a total wave propagating at a speed $v < c$. The key idea in understanding this reduced speed of light in the medium 8 is that the phase lag $\alpha$ of $E(x,t)$ must be linear in $x$ in order to achieve a constant phase velocity. This is due to the increasing contribution (with magnitude $E_1$ and delayed 90° in phase) from the growing region of polarized medium between the interface and $x$. The phase fronts of $E$ fall steadily farther behind those of $E_0$ as $x$ increases, implying a reduced constant phase velocity for the total wave $E$. When the light exits the medium, the polarized region (between the interface and $x$) stops growing, $\alpha$ becomes constant, and the phase velocity is again $c$.

The first-order perturbation treatment we have outlined is quantitatively inadequate for most common media, since $\chi_e$ is usually not $<< 1$. However, the mechanisms described (which lead to a continuously increasing phase lag of the transmitted wave) must still operate in the general case, providing at least a qualitative understanding of the reduction of the speed of light in dielectric media.

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3It appears that a negative sign should precede the expression on the right-hand side of Eq. (24) of Ref. 1.
4A derivation of this result by direct calculation of the vector potential appears in the Appendix of Ref. 1. Approximate treatments based on the radiation field of a single dipole are given in the textbook discussions of Ref. 2.
5I am grateful to the reviewer for clarifying this point.
6If the term dropped from Eq. (5) had been retained, $E_0$ in the denominator of Eq. (6) would be replaced by $E_0(1 - \chi_e/4)$, which clearly would change the result for $\alpha$ only by a term of second order in $\chi_e$. On the other hand, the dropped term does influence the magnitude of $E$ to first order, tending to reduce it. In the exact solution [Ref. 1, Eq. (17)] $E < E_0$, in contrast to what is pictured in the phasor diagram.
7Reference 1, Eq. (6).
8The following interpretation is based on the first-order approximation utilized here and the resulting phasor diagram of Fig. 1. To include effects of higher order terms, a more complicated analysis and diagram would be required.

Representing a vector field: Helmholtz's theorem derived from a Fourier identity

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In my undergraduate years, I learned that one could sometimes compute a vector field by forming the gradient of a scalar potential. Or maybe it was by forming the curl of a vector potential. Then again, perhaps I needed to form both. Although the textbooks that I read, such as Panofsky and Phillips' Classical Electricity and Magnetism, 1 did present a comprehensive discussion, I never saw the big picture, perhaps because the idea of various "sources" was so distracting.

Two years ago I happened upon a way to derive the essential theorem—Helmholtz's theorem—that seems to display the generality in a particularly compelling way. I outline the derivation here.

The context is a vector field $\mathbf{F}(x)$ defined over all of three-dimensional space: the entire infinite domain. The goal is to represent $\mathbf{F}(x)$ in some geometrically natural way; I will explain more about that in a moment.