Why the speed of light is reduced in a transparent medium

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It is well known from optics that the speed of light in a transparent medium is reduced by a factor of n (the index of refraction) as compared with vacuum. Maxwell’s electrodynamics provides a simple account of this phenomenon, and relates n to the electric susceptibility of the material. But the conventional analysis does little to illuminate the mechanism involved. This paper offers some elucidation of the “miracle” by which the radiation from many induced molecular dipoles conspires to produce a single wave propagating at the reduced speed.

I. INTRODUCTION

From Maxwell’s equations

\[ \mathbf{\nabla} \times \mathbf{E} = \varepsilon_0 \frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{\nabla} \times \mathbf{B} = \mathbf{\mu}_0 \mathbf{J} + \mathbf{\mu}_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \]

it follows that electromagnetic waves propagate through the vacuum at speed

\[ c = 1/\sqrt{\varepsilon_0 \mu_0}. \]

For if we apply the curl to Eq. (i) (ii), invoke the mathematical identity \( \mathbf{\nabla} \times (\mathbf{\nabla} \times \mathbf{E}) = \mathbf{\nabla}(\mathbf{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \), and set \( \rho = 0, \mathbf{J} = 0 \), we obtain the wave equation

\[ \nabla^2 \mathbf{E} = \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \]

with \( v = 1/\sqrt{\varepsilon_0 \mu_0} \).

In a linear dielectric medium of susceptibility \( \chi \), Maxwell’s equations can be written in the form

\[ \mathbf{\nabla} \times \mathbf{E} = (1/\varepsilon) \rho, \quad \mathbf{\nabla} \mathbf{B} = 0, \]

\[ \mathbf{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{\nabla} \times \mathbf{B} = \mathbf{\mu}_0 \mathbf{J}_f + \mathbf{\mu}_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \]

where \( \rho_f \) and \( \mathbf{J}_f \) are the free charge and current densities, and

\[ \varepsilon = (1 + \chi) \varepsilon_0. \]

The same argument as before, using \( \rho_f = 0 \) and \( \mathbf{J}_f = 0 \), leads again to the wave equation, but this time

\[ v = 1/\sqrt{\varepsilon_0 \mu_0} = c/\sqrt{1 + \chi}. \]

In this way classical electrodynamics accounts for the fact—familiar from geometrical optics—that the speed of light in a transparent medium is reduced by a factor of \( n \) (the index of refraction)—evidently

\[ n = 1 + \chi. \]

That argument is quick and beautiful, but it does little to elucidate the mechanism involved. Why should light travel slower in glass or water than in vacuum? Well, when an electromagnetic wave strikes dielectric material, the electric field induces an oscillating electric dipole in each molecule, and these oscillating dipoles radiate new electric and magnetic fields. By a marvelous coincidence, these secondary fields combine just right with the primary fields to produce a single wave propagating at the reduced velocity.

This story is perfectly correct, as far as it goes, but scarcely satisfying, since it relies on a seemingly miraculous conspiracy of the induced dipole fields. It becomes a little more plausible when reformulated as follows: an electric field \( \mathbf{E} \) produces a polarization \( \mathbf{P} \) in the material:

\[ \mathbf{P} = \varepsilon_0 \chi \mathbf{E}, \]

and a changing polarization creates a current \( \mathbf{J}_p \):

\[ \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \]

Inserting (8) and (9) into Maxwell’s equation (1) (iv) with total current \( \mathbf{J} = \mathbf{J}_f + \mathbf{J}_p \) yields

\[ \mathbf{\nabla} \times \mathbf{B} = \mathbf{\mu}_0 \mathbf{J}_f + \mathbf{\mu}_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \]

\[ = \mathbf{\mu}_0 \mathbf{J}_f + \mathbf{\mu}_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{\mu}_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t} \]

So the change from \( \varepsilon_0 \) to \( \varepsilon \) [between (1) and (4)] is indeed attributable to fields generated by currents associated with the polarization of the material.

Still, it would be nice to track the mechanism step by step, and see just how the “miraculous conspiracy” occurs. In this paper we study the process as a perturbation expansion in powers of \( \chi \). Imagine that a plane wave is incident on a piece of dielectric material. In zeroth order it simply continues along without modification. But this zeroth-order field polarizes the medium, and the resulting polarization currents give rise to a first-order field. This first-order field, in turn, further polarizes the medium, and the resulting currents generate a second-order field...and so on. We will show explicitly that the sum of all these fields is a wave propagating at speed \( c/n \) within the medium. Meanwhile, outside the dielectric the higher-order fields combine to form the reflected wave.

There are no surprises here—only a comforting confirmation that the story we have told is consistent, and perhaps a somewhat deeper understanding of the mechanism by which the speed of light is reduced in a dielectric medium. Before we begin—pursuant to Feynman’s famous injunction (never start a calculation until you know the answer)—we shall briefly review the standard (nonperturbative) approach to the problem.

II. TRANSMITTED AND REFLECTED WAVES AT A DIELECTRIC BOUNDARY

A monochromatic plane wave incident from vacuum \( (x < 0) \) on a transparent dielectric medium \( (x > 0) \) gives rise to a transmitted wave and a reflected wave:
For future reference we express these fields in terms of the susceptibility, using Eqs. (7) and (12):

\[ E_T(x,t) = \frac{2}{\sqrt{1 + \chi_e + 1}} E_0 e^{i(kx - \omega t)} j_e, \quad (x > 0), \]

\[ E_R(x,t) = - \left( \frac{\chi_e + 1}{\sqrt{1 + \chi_e + 1}} \right) E_0 e^{i(kx - \omega t)} j_e, \quad (x < 0). \]

In particular, expanding in powers of \( \chi_e \):

\[ E_T(x,t) = \left( \frac{2}{\sqrt{1 + \chi_e + 1}} \right) e^{i(kx - \omega t)} \sum_{n=0}^\infty \left( \frac{-ikx}{n!} \right)^n E_0 e^{i(kx - \omega t)} j_e \]

\[ = \left[ 1 - \frac{ikx}{\chi_e} (1 - 2ikx) + \frac{1}{2!} \chi_e^2 \right] \times (1 - 2ikx - k^2 x^2) + \cdots \]

\[ \times E_0 e^{i(kx - \omega t)} j_e. \]

\[ E_R(x,t) = - \frac{1}{\chi_e} \left( 1 - \frac{ikx}{\chi_e} + \frac{1}{2!} \chi_e^2 + \cdots \right) E_0 e^{i(kx - \omega t)} j_e. \]

### III. PERTURBATIVE APPROACH

We now attack the same problem from the perturbative perspective outlined in the Introduction. To zeroth order the incident wave continues on to the right:

\[ E^0(x,t) = E_0 e^{i(kx - \omega t)}, \quad (x > 0) \]  

[This is the first term in Eq. (18)], producing a polarization in the medium

\[ \mathbf{P}(x,t) = \varepsilon_0 \chi_e E^0 = \varepsilon_0 \chi_e E_0 e^{i(kx - \omega t)} j_e, \quad (x > 0). \]

The resulting polarization current is

\[ J_p(x,t) = \frac{\partial \mathbf{P}}{\partial t} = - i \omega \varepsilon_0 \chi_e E_0 e^{i(kx - \omega t)} j_e, \quad (x > 0). \]

To calculate the field generated by this polarization current we chop the dielectric into slabs of infinitesimal thickness \( dx' \) (see Fig. 2). In the Appendix we show that the electric field at a distance \( s \) from a neutral plane surface current \( \mathbf{K}(t) \) is

\[ \mathbf{E} = \mathcal{M} \frac{dx'}{2} \mathbf{K}(t - s/c). \]

So the field at \( x \) due to \( J_p \) is

\[ \mathbf{E}(x,t) = \left( \mathcal{M} \right) \frac{dx'}{2} \mathbf{K}(t - s/c). \]
\[ E'(x,t) = \left( -\frac{\mu c}{2} \right) \left( -i\omega e_0 \chi \hat{E}_0 \right) \]
\[ \times \left( \int_0^\infty e^{ikx - \alpha t - (x - x')/c} \, dx' + \int_x^\infty e^{ikx - \alpha t - (x - x')/c} \, dx' \right) \]
\[ = \frac{ik}{c} \chi_0 \hat{E}_0 \int_0^\infty e^{ikx - \alpha t} \, dx' + e^{i(kx - \alpha t)} \int_x^\infty e^{2ikx' - \alpha t} \, dx'. \]

which simplifies to

\[ E'(x,t) = \left( \chi_0/4 \right) (1 - 2ikx) E^0(x,t), \quad (x > 0) \quad (24) \]

[reproducing the second term in Eq. (18)].

This first-order field further polarizes the dielectric:

\[ P^2(x,t) = e_0 \chi_0 \hat{E}^2(x,t), \]

resulting in an additional polarization current

\[ J_p(x,t) = i\omega e_0 \chi_0 \hat{E}'(x,t), \]

which, in turn, generates the second-order field (calculated as before):

\[ E^2(x,t) = \left( \chi_0^2/2 \right) (1 - 2ikx - k^2 x^2) E^0(x,t), \quad (x > 0) \quad (25) \]

[consistent with the third term in Eq. (18)]. The progression is now clear: the nth-order field is of the form

\[ E^n(x,t) = (-\chi_0/2)^n Q_n(z) E^0(x,t), \quad (x > 0), \quad (26) \]

where \( Q_n \) is a polynomial of degree \( n \) in the variable \( z = -2ikx \). \( Q_0 = 1, \ Q_1 = (1 + z)/2, \ Q_2 = (1 + z + z^2/4)/2 \). The resulting current is

\[ J^{n+1}(x,t) = -i\omega e_0 \chi_0 \hat{E}^n(x,t), \quad (27) \]

and this produces a field

\[ J^{n+1}(x,t) = \frac{1}{2} (-\chi_0/2)^{n+1} E^0(x,t) \]
\[ \times \left( \int_0^\infty Q_n(z')dz' + e^\gamma \int_z^\infty Q_n(z') e^{-x} \, dz' \right). \quad (28) \]

Evidently the recursion relation for \( Q_n \) is

\[ Q_{n+1}(z) = \frac{1}{2} \left( \int_0^\infty Q_n(z')dz' + e^\gamma \int_z^\infty Q_n(z') e^{-x} \, dz' \right). \quad (29) \]

It follows that

\[ \frac{dQ_{n+1}}{dz} = \frac{1}{2} e^\gamma \int_z^\infty Q_n(z') e^{-x} \, dz' \]
\[ = Q_{n+1} - \frac{1}{2} \int_0^\infty Q_n(z')dz', \]

and hence

\[ \frac{d^2Q_{n+1}}{dz^2} - \frac{dQ_{n+1}}{dz} + \frac{1}{2} Q_n = 0. \quad (30) \]

Now, the total transmitted wave is

\[ E^P(x,t) = \sum_{n=0}^\infty E^n(x,t) = f(z) E^0(x,t), \quad (x > 0), \quad (31) \]

where

\[ f(z) = \sum_{n=0}^\infty \left( -\frac{X_0}{2} \right)^n Q_n(z). \quad (32) \]

From Eq. (30) it follows that

\[ \frac{d^2f}{dz^2} - \frac{df}{dz} + \frac{1}{2} \sum_{n=0}^\infty \left( -\frac{X_0}{2} \right)^n Q_{n-1} = 0, \]

or, redefining the summation index \( n \rightarrow n+1 \):

\[ \frac{d^2f}{dz^2} - \frac{df}{dz} - \frac{X_0}{4} f = 0. \quad (33) \]

This is a differential equation for \( f(z) \); the general solution is

\[ f(z) = A(\chi_0) e^{(1 - \sqrt{1 + \chi_0}) z^2/2} + B(\chi_0) e^{(1 + \sqrt{1 + \chi_0}) z^2/2}. \quad (34) \]

It remains to determine \( A(\chi_0) \) and \( B(\chi_0) \).

For \( \chi_0 = 0 \) we get \( f(z) = A(0) + B(0) e^z \); however, from (32) we know that \( f(z) = Q_0(z) = 1 \) when \( \chi_0 = 0 \). Evidently \( A(0) = 1 \) and \( B(0) = 0 \). Similarly, differentiating (34) with respect to \( \chi_0 \) and setting \( \chi_0 = 0 \) we find \( f'(z) = A'(0) - z/4 + B'(0) e^z \), whereas (32) gives \( f'(z) = -iQ_1(z) = -1/4 - z/4 \). So \( A'(0) = -1/4 \) and \( B'(0) = 0 \). As we continue in this way the derivatives of \( B \) will always be accompanied by a factor of \( e^z \), which is absent in (32), and hence all the derivatives of \( B \) must vanish.

Conclusion:

\[ B(\chi_0) = 0. \quad (35) \]

To determine \( A(\chi_0) \) we proceed as follows: From (29) we have

\[ Q_{n+1}(0) = \frac{1}{2} \int_0^\infty e^{-x} Q_n(z')dz', \quad \text{for } n > 0. \quad (36) \]

So (32) yields

\[ f(0) = \sum_{n=0}^\infty \left( -\frac{X_0}{2} \right)^n Q_n(0) \]
\[ = Q_0(0) + \sum_{n=1}^\infty \left( -\frac{X_0}{2} \right)^n \int_0^\infty e^{-x} Q_{n-1}(z')dz' \]
\[ = 1 - X_0 \int_0^\infty e^{-x} \left( \sum_{n=1}^\infty \left( -\frac{X_0}{2} \right)^n Q_{n-1}(z')dz' \right) \]
\[ = 1 - X_0 \int_0^\infty e^{-x} f(z')dz'. \quad (37) \]

Inserting (34), with \( B = 0 \), we have

\[ A(\chi_0) = 1 - X_0 \int_0^\infty e^{-x} e^{(1 - \sqrt{1 + \chi_0}) z^2/2} dz' \]
\[ = 1 - X_0 \int_0^\infty e^{-x} e^{2/(1 + \sqrt{1 + \chi_0})} dz' \]
\[ = \frac{2}{1 + \sqrt{1 + \chi_0}}. \quad (38) \]

Or, solving for \( A \):

\[ A(\chi_0) = 2/(1 + \sqrt{1 + \chi_0}), \quad (38) \]

and hence

\[ f(z) = \left[ 2/(1 + \sqrt{1 + \chi_0}) \right] e^{(1 - \sqrt{1 + \chi_0}) z^2/2}. \quad (39) \]

Returning, finally, to Eq. (31)—and recalling that \( z = -2ikx \)—we conclude that

\[ f(z) = \sum_{n=0}^\infty \left( -\frac{X_0}{2} \right)^n Q_n(z). \quad (32) \]
\[ E^r(x,t) = \frac{2}{1 + \sqrt{1 + \chi_e^2}} e^{-i\kappa(x + c_0 t)} E_{0e} e^{i(kx - \omega t)} j \]

\[ = \frac{2}{1 + \sqrt{1 + \chi_e^2}} E_{0e} e^{i(\kappa x - \omega t)j} \]

which is precisely what we got in Sec. II [Eq. (17)] by more traditional means.

IV. THE REFLECTED WAVE

The reflected wave (in the region \( x < 0 \)) can be recovered by the same procedure. In zeroth order there is no reflection; to first order we have the field generated by \( J_p \) [Eq. (22)]:

\[ E^r(x,t) = \left( -\frac{\mu_0 e}{2} \right) \left( -i\omega e_0 \chi_e E_0 j \right) \int_0^\infty e^{i(kx - \omega t - (x' - x)/c)} dx' \]

\[ = \frac{i\kappa}{2} \chi_e E_{0e} e^{i(-kx - \omega t)} j \]

\[ = \frac{-\chi_e}{4} E_{0e} e^{i(-kx - \omega t)} j \]

which is consistent with the first term in Eq. (19). The second-order field is generated by \( J_p^2 \):

\[ E^2(x,t) = \left( -\frac{\mu_0 e}{2} \right) \left( -i\omega e_0 \chi_e \left( -\frac{\chi_e}{4} E_0 j \right) \right) \]

\[ \times \int_0^\infty (1 - 2ikx') e^{i(kx - \omega t - (x' - x)/c)} dx' \]

\[ = \left( \frac{-i\kappa \chi_e^2}{8} \right) E_{0e} e^{i(-kx - \omega t)} j \]

\[ \times \int_0^\infty (1 - 2ikx') e^{2ikx'} dx' \]

\[ = \frac{\chi_e^2}{8} E_{0e} e^{i(-kx - \omega t)} j \]

consistent with the second term in Eq. (19).

In general, \( E^{n+1} \) is generated by \( J_p^{n+1} \) [Eq. (27)]:

\[ E^{n+1}(x,t) = \left( -\mu_0 e/2 \right) \left( -i\omega e_0 \chi_e \right) \left( -\frac{\chi_e}{2} \right)^n E_0 \]

\[ \times e^{i(-kx - \omega t)} j \left( \frac{1}{2ik} \right) \int_0^\infty Q_n(x') e^{-z} dz' \]

\[ = E_{0e} e^{i(-kx - \omega t)} j \left( \frac{-\chi_e}{2} \right)^{n+1} \frac{1}{2} \]

\[ \times \int_0^\infty Q_n(x') e^{-z} dz' \]

\[ = E_{0e} e^{i(-kx - \omega t)} j (\chi_e/2)^{n+1} Q_{n+1}(0) \]

[we used Eq. (36) in the last step]. Thus the total reflected wave is

\[ E^R(x,t) = \sum_{n=1}^{\infty} E^n(x,t) \]

\[ = \left( \sum_{n=1}^{\infty} \left( \frac{-\chi_e}{2} \right)^n \right) E_{0e} e^{i(-kx - \omega t)} j \]

\[ = \left( \frac{1 - \sqrt{1 - \chi_e^2}}{1 + \sqrt{1 + \chi_e^2}} \right) E_{0e} e^{i(-kx - \omega t)} j \]

which is what we got in Sec. II [Eq. (17)] by the conventional method.

V. CONCLUSION

We do not, of course, pretend that the perturbative approach developed here is superior to the traditional one. But it is instructive (and comforting) to see in detail how the induced polarization currents in a dielectric medium give rise both to a reflected wave and to a transmitted wave traveling at the reduced speed \( c/n \).

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APPENDIX: THE ELECTRIC FIELD OF A NEUTRAL PLANE CURRENT SHEET

Suppose that the \( yz \) plane carries a uniform (but time-dependent) surface current \( K(t) \). The retarded vector potential at a point a distance \( x \) above the plane is given by

\[ A(x,t) = \frac{\mu_0}{4\pi} \int \frac{K(t_0)}{\sqrt{r^2 + x^2}} da, \]

where \( t_0 = t - \sqrt{r^2 + x^2}/c \) is the retarded time, and \( da = 2\pi dr \) is an infinitesimal element of area (see Fig. 3).

Thus

\[ A(x,t) = \frac{\mu_0}{2} \int_0^\infty \frac{K(t - \sqrt{r^2 + x^2}/c)}{\sqrt{r^2 + x^2}} dr \]

To simplify the integral, let \( u = (1/c)(\sqrt{r^2 + x^2} - x) \), so that

![Fig. 3. The electric field of a current sheet.](image-url)
du = \frac{1}{c} \frac{r}{\sqrt{r^2 + x^2}} dr \quad \text{and} \quad t - \frac{\sqrt{r^2 + x^2}}{c} = t - \frac{x}{c} - u.

Then

\[ A(x,t) = \frac{\mu_0 c}{2} \int_0^\infty K \left( t - \frac{x}{c} - u \right) du. \]  

(A3)

From the vector potential it is easy to compute the electric field: 2

\[ E(x,t) = \frac{\partial A}{\partial t} = -\frac{\mu_0 c}{2} \int_0^\infty \left[ \frac{\partial}{\partial t} K \left( t - \frac{x}{c} - u \right) \right] du \\
= \frac{\mu_0 c}{2} \int_0^\infty \left[ \frac{\partial K}{\partial u} \right] \left( t - \frac{x}{c} - u \right) du \\
= \frac{\mu_0 c}{2} \left[ K \left( t - \frac{x}{c} - u \right) \right]_{u=0}^{u=\infty} \\
= -\frac{\mu_0 c}{2} \left[ K \left( t - \frac{x}{c} \right) - K(-\infty) \right].

Assuming the current goes to zero in the distant past, we can drop the second term, leaving

\[ E(x,t) = -\frac{\mu_0 c}{2} K \left( t - \frac{x}{c} \right). \]  

(A4)

2Reference 1, Chap. 30.
3Equation (21) presupposes that the current goes to zero in the distant past (see the Appendix). Although this is not strictly true for the sinusoidal current in Eq. (22), it better describes the physical circumstance, in which the incident light source was turned on at some finite time. As a formal device we can handle the problem by attaching a small imaginary part to the frequency \((\omega - i \epsilon)\), which attenuates the field at large negative \(t\), and taking the limit \(\epsilon \to 0\) at the end of the calculation.
4The upper limit on the second integral yields a term of the form \(e^{i(kx - ct - \epsilon)}\) (with \(L \to \infty\)), representing a wave reflected back from the far side of the dielectric (at \(x = L\)). Obvioulsy, we do not want such a term here; it is, again, an artifact of the pure sinusoidal motion (22), and disappears if we attenuate the incident wave, as in Ref. 3: the factor \(e^{i\omega t}\) then goes to zero in the limit \(L \to \infty\).
5Actually, the upper limit on the second integral is \(2L(\epsilon/c - ik)\), with \(L \to \infty\). But the integrand is nonsingular, and a suitable rotation in the complex plane brings it around to the real axis.
6Reference 1, Sec. 37.4.
7Reference 1, Eq. 23.46. Note that since the surface is neutral, the scalar potential is zero.

Thomas precession: Where is the torque?

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Special relativity appears to violate the conservation of angular momentum \(L\) since it predicts that an accelerated gyroscope will precess, i.e., \(L\) will change in the absence of any applied torque. The paradox is resolved in a simple example by demonstrating that there is a torque present. The mass distribution in the gyroscope undergoes a relativistic distortion, and the center of mass is displaced away from the position of the accelerating force. The resulting torque \(\tau = dL/dt\). The model also shows the physical origins of spin-orbit coupling and of the “oscillating term.” A related calculation shows why a moving magnetic dipole has an electric dipole moment.

I. INTRODUCTION

According to the special theory of relativity, a gyroscope that moves along a curved path will also precess, i.e., the direction of its spin angular momentum will change. This effect is known as the “Thomas precession” after L. H. Thomas, who showed how this effect could resolve a paradox in atomic physics. 1 The standard derivation 2 of Thomas precession uses the fact that the product of successive Lorentz transformations is equivalent to a single Lorentz transformation plus a rotation. The most straightforward derivation known to the author is reproduced in the Appendix.

Even those who feel comfortable with one of the standard derivations may be at a loss to explain how angular momentum can be conserved. Angular momentum \(L\) is related to torque \(\tau\) through the vector relation

\[ \frac{dL}{dt} = \tau. \]

Yet the angular momentum of the gyroscope is changing, in the apparent absence of torque. How can that happen?

The resolution of the paradox is simple: There is a torque applied to the gyroscope, by the same force that accelerates the gyroscope along the curved path. The torque exists because of a relativistic distortion of the mass distribution in the gyroscope that moves the center of mass away from the axle.