Mathematical Apparatus of
THE THEORY OF
ANGULAR MOMENTUM

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FOREWORD

The principal results obtained up to 1935 in the quantum-mechanical theory
of angular momentum are contained in chapter III of Condon and Shortley’s “Theory
of Atomic Spectra” /1949/. Since then, owing to the ideas of Wigner /1931,
1937/ and Racah /1942/, the theory has been enriched by the algebra of noncom-
muting tensor operators and the theory of j-coefficients. This has considerably in-
creased its computational possibilities and has broadened the scope of its applications.
Among the branches of theoretical physics where the methods of the theory of angular
momentum are widely applied today we might mention the theory of atomic and
nuclear spectra, the scattering of polarized particles in nuclear reactions, the theory
of genealogical coefficients, etc. (a bibliography of the applications may be found in
Edmonds’ book /1957/).

The only book known to us giving an exposition of the algebra of noncommuting
tensor operators and j-coefficients is Edmonds’ “Angular Momentum in Quantum Me-
chanics” /1957/, which may serve as an excellent textbook for a first acquaintance
with the subject. However, the exposition of the theory of j-coefficients and trans-
formation matrices given in this book is not complete. This may constitute an impe-
diment when the apparatus is employed in more complicated cases. The present
book fills this gap.

The writing of this book began before Edmonds’ book appeared in print. The
authors have utilized nearly all results known to them in the given field. Among these
a certain place is occupied by the results obtained by a group of workers under the
direction of one of the present authors (H. Yutis), the remaining two authors
(I. Levinson and V. Vanagas) being the principal participants. The book corresponds
to the content of the first part of a course, “Methods of Quantum-Mechanical Atomic
Calculations”, given by the senior author to students of theoretical physics at the
Vilnius University (V. Kupcenas) over the last two years.

We found it worthwhile to use the elegant and powerful methods of group theory
in our exposition. To avoid encumbering the book with elements of group theory we
have assumed that the reader is already acquainted with linear representations of the
three-dimensional rotation group. The reader who is unfamiliar with this may refer
to the books by G. Ya. Lyubashen /1957/ and L. M. Gel’fand et al. /1958/.

We begin with the well-known theory of vector addition of two angular momenta
(chapter II), turning next to the addition of an arbitrary number of angular momenta
(chapter III). The following chapters (III-VI) are devoted to quantities of the theory

* English translation by Stevan Dedijer, Pergamon Press, 1960; alternatively, the
reader may consult Wigner /1953/, or Racah /1951/ – Translator.
of angular momentum where an important place is occupied by the graphical method which is convenient for various calculations. The last chapter (VII) deals with the method of noncommuting tensor operators. Material of a supplementary character is given in the appendices.

We have cited a number of unpublished works some of which were not available to us. References to these were based on other published works. We apologize in advance for any resulting inaccuracy.

The authors will be grateful for criticism which should be addressed to:
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Vilnius
August, 1959

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TRANSLATOR’S NOTE

The present book is the translation of a Russian work published in Vilnius and carrying an alternative title page in the Lithuanian language, which is written in Latin script. The translators were therefore faced with an interesting and nontrivial problem, that of deciding whether to spell the names of the authors as they are written in Lithuanian, or as they would be transliterated from the Russian. The latter alternative was adopted as being the form which would be most easily recognizable in the literature in Russian; to the best of our knowledge, a large number of works of the authors of this book have been published in Russian, and the Western reader is most likely to encounter references to these in Russian periodicals. The Lithuanian version of the authors’ names, as given on the Lithuanian title page, is:

A. Jacys
J. Levinsonas
V. Vanagas

This system has led to a certain double-valuedness within the translation itself. The reader will find both “Yutis” and “Jacys” in the text and in the bibliography. The first is used to quote references cited in Russian, and the second for references cited in Lithuanian.

The authors’ address at the end of the foreword was given only in Cyrillic; we were therefore obliged to render it as a transliteration from the Russian. The only exception is the place-name Vilnius, which in transliteration would be Vil’nyus and is known to many as Vils.

If the reader opens the book at random, he is likely to get another surprise, e.g., “Condon and Shortley 1949”. A large number of standard Western works seem to be in circulation in the USSR in Russian translation. Several such translations are quoted in the present book. We have in such cases stuck to the book and have referred to the Russian translation. In the bibliography we have also given the original reference, not because any reader of this book will need to be reminded of it, but out of a desire for completeness.

In matters of terminology we have occasionally departed from the Russian original. We hope that the reader of this translation will not “feel” it, which we also put forward as our justification for taking this liberty. We have tried to render the scientific content of the book with the utmost faithfulness. However, it is well known that translations, unlike rotations, cannot always be represented in a “unitary” form. We therefore do not guarantee that an independent translation of the present book into Russian will recover the original.
Chapter I

ADDITION OF TWO ANGULAR MOMENTA

In this introductory chapter we shall deal with angular momentum and its properties, which is the basis for subsequent development of the mathematical apparatus of this book. Section 1 sketches the connection between angular momentum and the three-dimensional rotation group. Section 2 deals with the corresponding eigenfunctions and representations. The addition of two angular momenta and its relation to the reduction of the direct product of two representations is given in section 3. The Clebsch-Gordan coefficients are examined in section 4, and finally in section 5 the more symmetric Wigner coefficients are introduced.

1. Angular momentum operators and spatial rotations

Let the function \( \psi \) describe the state of a physical system in a certain system of cartesian coordinates. If we transform to a new system of coordinates, obtained by rotating the old system through an angle \( \omega \) about the axis \( \pi \), the state of the physical system will then be described by a function \( \psi' \) which may be obtained from \( \psi \) by the operation of an unitary operator \( V(\pi, \omega) \); the latter of course depends on the parameters of the rotation

\[
\psi' = V(\pi, \omega) \psi.
\]  

(1.1)

For \( \omega = 0 \), \( V \) will obviously be the identity operator.

Let us write the operator \( V \) in the form

\[
V(\pi, \omega) = e^{-iJ(\pi, \omega)}.
\]  

(1.2)

The operator \( J \) must tend to zero with \( \omega \), and for small \( \omega \) we therefore have

\[
J(\pi, \omega) = J_\pi \omega.
\]  

(1.3)

Inserting (1.3) into (1.2) and expanding the exponential in series we obtain, for small \( \omega \),

\[
V(\pi, \omega) = I - i \omega J_\pi.
\]  

(1.4)

where \( I \) denotes the identity operator. For small \( \omega \) (1.1) therefore becomes

\[
\delta \psi = \psi' - \psi = -i \omega J_\pi \psi.
\]  

(1.5)

Thus, up to the factor \(-i\), the operator \( \omega J_\pi \) gives the change in the function \( \psi \) upon an infinitesimal rotation about the axis \( \pi \).
From the unitarity of \( V \) (\( V^{\dagger}V = 1 \), where \( V^{\dagger} \) is the hermitian conjugate of \( V \)) it follows that the operator \( J_{z} \) is hermitian. Clearly, from (1.4) we have
\[
(1 + i a J_{z}^{a})(1 - i a J_{z}^{a}) = 1,
\]
whence it follows that \( J_{z}^{a} = J_{z}^{a} \) (condition of hermiticity). The operator \(-i J_{z}^{a}\) is the operator of infinitesimal rotation about the \( z \) axis.

The three operators \( J_{x}, J_{y}, J_{z} \) of infinitesimal rotations about the cartesian axes satisfy the well-known commutation relations (Condon and Shortley 1949/)
\[
[J_{x}, J_{y}] = i J_{z}, \quad [J_{y}, J_{z}] = i J_{x}, \quad [J_{z}, J_{x}] = i J_{y},
\]
where \([a, b] \) denotes the commutator \( ab - ba \).

The commutation relations (1.6) are general conditions which are satisfied by the components of an arbitrary angular momentum (measured in units of \( h/2\pi \)). In particular, they include both the orbital angular momentum \( L \) with components
\[
L_{x} = -i \left( \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \phi} \right), \quad \text{and analogously for} \quad L_{y} \quad \text{and} \quad L_{z},
\]
and the spin angular momentum with components expressed by the Pauli matrices
\[
S_{x} = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}, \quad S_{y} = \begin{pmatrix} 0 & -i \\ i/2 & 0 \end{pmatrix}, \quad S_{z} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.
\]

In the first case, the function \( \psi \) which describes the state of the system is a function of the space coordinates; in the second case, it is a spinor with two rows and one column.

2. Angular momentum eigenfunctions and representations of the rotation group

We shall use the term "angular momentum eigenfunctions" to denote the eigenfunctions of the operators \( J^{2} \) and \( J_{z} \). From the commutation relations (1.6) it may be shown (Condon and Shortley 1929/1) that the eigenvalues of the operator \( J^{2} \) are \( j(j + 1) \), where \( j = 0, 1/2, 1, 3/2, \ldots \), and that the eigenvalues of \( J_{z} \) are \( m = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots \). The eigenfunctions of the angular momentum \( J \) may therefore be denoted by \( \psi(jm) \). The quantities \( j \) and \( m \) are either both integral or both half-integral. Further, \( |m| \leq j \) and for given \( j, m \) may assume \( 2j + 1 \) values. \( j \) is usually called the angular momentum quantum number and \( m \) its projection on the \( z \) axis or the magnetic quantum number. Operating on the eigenfunctions, \( J^{2} \) and \( J_{z} \) give
\[
J^{2}\psi(jm) = j(j + 1) \psi(jm),
\]
\[
J_{z}\psi(jm) = m \psi(jm),
\]
\[
(J_{x} \pm iJ_{y})\psi(jm) = [J_{x} \pm iJ_{y}](j \pm m)(j \pm m + 1)^{1/2} \psi(jm \pm 1).
\]

These equations determine the standard choice of phases for wave functions with different \( m \).

The last two equations in (2.1) show that, for fixed \( j \), the functions \( \psi(jm) \) transform among each other under infinitesimal rotations. Hence these functions transform among each other under finite rotation:
\[
V(g)\psi(jm) = \sum_{m'} \psi(j'm') |m'\rangle \langle D_{j}(g)|m',
\]
(2.2)

Here \( g \) denotes an arbitrary rotation and \( V(g) \) the operator of this rotation. The equation (2.2) means that the transformed functions are expressed as linear combinations of the original functions \( \psi(jm) \). The coefficients of these linear combinations are elements of the matrix \( D_{j}(g) \) of order \( 2j + 1 \) which depends on the parameters of the rotation \( g \). The matrices \( D_{j}(g) \) with different \( g \) form an irreducible representation \( D_{j} \) of the rotation group (see, for instance, Wigner 1931/, Gel'fand et al. 1958/, Lyubarskii 1957/1), i.e.,
\[
D_{j}(g_{1}) \cdot D_{j}(g_{2}) = D_{j}(g_{1} \cdot g_{2}),
\]
(2.3)
or, in terms of the matrix elements,
\[
\sum_{m'} \langle m | D_{j}(g_{1}) | m' \rangle \langle m' | D_{j}(g_{2}) | m'' \rangle =
\sum_{m'} \langle m | D_{j}(g_{1} \cdot g_{2}) | m'' \rangle =
\]
(2.3a)

In accordance with (2.2), the functions \( \psi(jm) \) form a basis for the representation \( D_{j} \).

The representation \( D_{j} \) is unitary:
\[
D_{j}(g^{-1}) = D_{j}^{-1}(g) = D_{j}^{*}(g),
\]
(2.4)
or, which is the same,
\[
\langle m | D_{j}(g^{-1}) | m' \rangle = \langle m | D_{j}(g^{-1}) | m' \rangle = \langle m | D_{j}(g) | m \rangle^{*}.
\]
(2.4a)

The condition of unitarity may also be written as
\[
D_{j}(g) D_{j}^{*} = 1,
\]
(2.5)
or, in terms of the matrix elements
\[
\sum_{m} \langle m | D_{j}(g) | m' \rangle \langle m' | D_{j}(g) | m'' \rangle = \delta(m, m'').
\]
(2.5a)

Further, we have the relation
\[
\langle m | D_{j}(g) | m'' \rangle^{*} = (-1)^{j-m} \langle -m | D_{j}(g) | -m'' \rangle =
\]
(2.5)
This property of the matrices \( D_j \) is related to the following property of the functions \( \psi \):

\[
\psi^* (j - m) = (-1)^{j - m} \psi (j - m). \tag{2.7}
\]

Using (2.6), the condition of unitarity may be written in yet another form:

\[
\sum_{m'} (-1)^{j - m'} \langle m' | D_j (g) | m \rangle \langle m' | D_j (g) | m \rangle^* = \delta (m, m'). \tag{2.5b}
\]

The phase factor \((-1)^{j - m'}\) may, if desired, be written in the more symmetrical form \((-1)^{j - m'} (j - m')\). The matrices \( D_j (g) \) form a system of functions in \( g \), orthogonal on the group of elements \( g \):

\[
\int \left( m' | D_j (g) | m \right) \left( m' | D_j (g) | m \right)^* \frac{dg}{dG} = \delta (j, j') \delta (m, m') \times \delta (m', m) \times (-1)^{j - m} \delta (m', m). \tag{2.6}
\]

Here \( G \) is the volume of the rotation group. Noting that \( (0 | D_j (g) | 0) = 1 \), we obtain from (2.6)

\[
\int \left( m | D_j (g) | m \right) \left( m | D_j (g) | m \right)^* \frac{dg}{dG} = \delta (j, 0) \delta (m, 0) \delta (m', 0). \tag{2.7}
\]

Here we should add a few remarks concerning the matrices \( D_j \) with half-integral \( j \). These matrices do not form representations of the group in the strict sense of the word, as to each rotation \( g \) there corresponds two matrices \( D_j \) with different signs. This is related to the fact that the wave functions of systems with nonintegral spins change sign upon rotation by \( 2\pi \) about an arbitrary axis. One may avoid this two-valuedness and restrict oneself to usual single-valued representations by formally introducing the so-called double rotation group /Beth 1929/. In the double rotation group a rotation through \( 2\pi \) about an arbitrary axis is regarded as an element \( g \) which is different from the unit element. As a result, the number of elements of the group and the group volume are doubled. When the double rotation group is introduced the matrices \( D_j \) with integral and with half-integral \( j \) form the usual representations of this group. In all formulas in this section \( g \) should be understood as an element of the double rotation group, only in which case will the formulas assume a rigorous meaning for half-integral \( j \). Instead of the double rotation group one can consider a group which is isomorphic to it, namely the group of unimodular matrices:

\[
u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \text{Det} \nu = aa^* + bb^* = 1. \tag{2.10}
\]

In this point the reader may consult Wigner /1931/, chapter 15 (the section on "The Homomorphism of the Two-Dimensional Unitary Group onto the Rotation Group", p. 157 of the English translation), or "Topological Groups", by Leon Pontrjagin, Princeton University Press, 1939, ch. IX, for a more general discussion.

The explicit form of the matrices \( D_j \), which are expressed in terms of the Euler angles or the Cayley-Klein parameters \( a \) and \( b \), was obtained by Wigner /1931/. The first of these expressions can only be used for the usual group of rotations and integral \( j \). The second is also suitable for the double rotation group and for half-integral \( j \). The asymptotic expressions for the matrices \( D_j \) for large \( j \) (semi-classical approximation) can be found in Brussard and Tolhoek /1957/.

3. Addition of angular momenta: reduction of the direct product of representations of the rotation group.

The wave functions of a quantum-mechanical system in a spherically symmetric field are the eigenfunctions of a certain angular momentum \( j \) and are characterized by the quantum numbers \( j \) and \( m \). We shall denote these functions by \( \psi (j, m) \), where \( a \) is the set of additional quantum numbers required to complete the set. We note that if a different set of additional quantum numbers \( b \) is chosen, the transformation matrix will have the following property /Condon and Shortley 1949/:

\[
(a | j, m; b | j', m') = \delta (j, j') \delta (m, m') (a | b). \tag{3.1}
\]

In the absence of an external field, the energy of the system does not depend on \( m \) (i.e., on the orientation of the "vector" \( j \) in space), and therefore all wave functions with the same \( a \) and \( j \) belong to the same energy level, which thus has a \((2j + 1)\)-fold degeneracy.

Let there be two weakly interacting systems. In the zeroth approximation, i.e., when the interaction between the systems is neglected, the wave function of the coupled system is the product of the wave functions of its component parts

\[
\psi (a_1 j_1 a_2 j_2; m_1 m_2) = \psi (a_1 j_1 m_1) \psi (a_2 j_2 m_2). \tag{3.2}
\]

and the energy of the coupled system, being the sum of energies of the component parts, depends only on \( a_1 j_1 a_2 j_2 \). Consequently, all the \((2j_1 + 1)(2j_2 + 1)\) wave functions (3.2) with different \( m_1 \) and \( m_2 \) belong to the same energy level, which is \(j\)-fold degenerate. To calculate the interaction in the first approximation we must, as is known, set up the proper wave functions \( \psi \) in the zeroth approximation which are linear combinations of the wave functions \( \psi \) of the zeroth approximation belonging to the same energy level. The proper wave functions in the zeroth approximation should be the eigenfunctions of the operator \( j = j_1 + j_2 \) with certain quantum numbers \( j \) and \( m \). This requirement can be satisfied by choosing the coefficients of the linear combination to be independent of \( a_1 \) and \( a_2 \). Thus we have

\[
\Psi (a_1 j_1 a_2 j_2 m_1 m_2) = \sum_{m_1} \psi (a_1 j_1 m_1) \psi (a_2 j_2 m_2) (j_1 j_2 m_1 m_2 | j_1 j_2 m_2). \tag{3.3}
\]

The construction of eigenfunctions of the operators \( j = j_1 + j_2 \) from the eigen-functions of the operators \( j_1 \) and \( j_2 \) described by the equation (3.3) will be
termed the addition or coupling of the angular moments \( j_1 \) and \( j_2 \) to the angular momentum \( j \), and the wave function (3.3) the wave function of two coupled angular momenta. Further, we assume that the eigenfunctions of each angular momentum do not undergo transformations of the type (3.1) upon coupling. In such cases we shall simply omit the quantum numbers \( a_1 \) and \( a_2 \). If the coupling is accompanied by a transformation of the eigenfunctions of the individual angular momenta, then, since from (3.1) the matrices of these transformations are independent of the magnetic quantum numbers, we have

\[
(a_1 j_1 a_2 j_2 m_1 m_2 | \beta_1 j_1 \beta_2 j_2 j_1 j_2 j m) = (a_1 j_1 | \beta_1 j_1) (a_2 j_2 | \beta_2 j_2) (j_1 j_2 m_1 m_2 | j_1 j_2 j m).
\]  

(3.4)

This equation shows that the transformations of the wave functions of the individual angular momenta are independent of the coupling. The inverse transformation from the functions (3.3) to the functions (3.4) is given by

\[
\psi (a_1 j_1 m_1) \psi (a_2 j_2 m_2) = \sum_{j m} \psi (a_1 j_1 a_2 j_2 j m) (j_1 j_2 m_1 j_1 j_2 m_2).
\]  

(3.5)

The coupling of angular momenta can be regarded as the transformation of an orthonormal system of functions \( \psi (m m') \) belonging to a given energy level into another orthonormal system \( \psi (j m) \) belonging to the same level. The coefficients of the expansion (3.3) should then be understood as the elements \( (m m' | C_{j m} j m) \) of an unitary matrix \( C_{j m} \), which effects this transformation. It is obvious that the coefficients of the expansion (3.5) will then be the elements \( (j m | C_{j m}^{-1} j m) \) of the inverse matrix \( C_{j m}^{-1} \). The matrix \( C \) may be chosen to be real. The coefficients of the expansions (3.3) and (3.5) will then be identical. This means that

\[
(j_1 j_2 m_1 m_2 | j_1 j_2 j m) = (j_1 j_2 m_1 m_2 | j_1 j_2 m_1 m_2).
\]  

(3.6)

These coefficients are called the vector addition or Clebsch-Gordan coefficients.

The conditions of unitarity of the matrix \( C \) reduce to the following orthogonality conditions of the Clebsch-Gordan coefficients

\[
\sum_{m m'} (j_1 j_2 m_1 m_2 | j_1 j_2 m_1 m_2) (j_1 j_2 m_1 m_2 | j_1 j_2 m_1 m_2) = \delta (j_1 j_2) \delta (m m'),
\]  

(3.7a)

\[
\sum_{j m} (j_1 j_2 m_1 m_2 | j_1 j_2 m_1 m_2) (j_1 j_2 m_1 m_2 | j_1 j_2 m_1 m_2) = \delta (m_1 m_1) \delta (m_2 m_2).
\]  

(3.7b)

The Clebsch-Gordan coefficients are non-zero only when the following conditions are fulfilled

\[
j_1 + j_2 \geq j \geq |j_1 - j_2|,
\]  

(3.8a)

\[
j_1 + j_2 + j = \text{integer},
\]  

(3.8b)

\[
m_1 + m_2 = m.
\]  

(3.8c)

These are the so-called integral perimeter triangle conditions or simply the triangle conditions.

The Clebsch-Gordan coefficients are closely related to the representations of the rotation group. The \( (2j_1 + 1)(2j_2 + 1) \) functions (3.2), in which \( j_1 \) and \( j_2 \) are independent, form the basis of the direct (Kronecker) product representation \( D_{j_1} \times D_{j_2} \) of the rotation group. In fact

\[
\psi (j_1 a_1 j_2 a_2 j_1 j_2 j m) = \sum_{m_n} \psi (j_1 a_1 j_2 a_2 j_1 j_2 j m) (m_n m_n | D_{j_1} (g) \times D_{j_2} (g)) (m_n m_n),
\]  

(3.9)

where the matrix elements of the direct product are

\[
(m_n m_n | D_{j_1} (g) \times D_{j_2} (g)) (m_n m_n) = (m'_n | D_{j_1} (g)) (m'_n | D_{j_2} (g)) (m_n m_n).
\]  

(3.10)

The direct product \( D_{j_1} \times D_{j_2} \) is reducible; this means that there exists a matrix \( C_{j j} \) such that

\[
C_{j j}^{-1} (D_{j_1} \times D_{j_2}) C_{j j} = D_{j_1}^{j_1} \times \cdots \times D_{j_2}^{j_2}. \]

(3.11)

Here the right-hand side contains a 'quasi-diagonal' matrix with the submatrices \( D_{j_1} \), \( j = j_1 + j_2, \ldots, |j_1 - j_2| \); further, each submatrix \( D_{j_2} \) appears only once. The wave functions of the coupled angular momenta form the basis of the reduced representation, and the matrix \( C_{j j} \) is identical with the above-mentioned matrix. The Clebsch-Gordan coefficients are thus the elements of the matrix which reduces the direct product of two irreducible representations of the rotation group /Wigner 1931/. In terms of the matrix elements equation (3.11) therefore assumes the form

\[
\sum_{m m'} (j_1 j_2 m_1 m_2 m_3 m_4 m_5 m_6) (m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6) \times
\]

\[
\sum_{m m'} (j_1 j_2 m_1 m_2 m_3 m_4 m_5 m_6) (m_1 m_2 m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6).
\]  

(3.11a)

From the unitarity of the Clebsch-Gordan coefficients (3.7), we obtain the inverse relation

\[
(m_1 m_2 m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6) = \sum_{m m'} (j_1 j_2 m_1 m_2 m_3 m_4 m_5 m_6) (m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6).
\]  

(3.11b)

Multiplying this equation by \( (m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6) \) and integrating over the group according to (2.8) we obtain the relation

\[
\int (m_1 m_2 m_3 m_4 m_5 m_6) (m_1 m_2 m_3 m_4 m_5 m_6) (m_1 m_2 m_3 m_4 m_5 m_6) (m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6) d^3 g =
\]

\[
= (-1)^{j_1 + j_2} (2j_1 + 1)^{-1} (j_1 j_2 m_1 m_2 m_3 m_4 m_5 m_6) (j_1 j_2 m_1 m_2 m_3 m_4 m_5 m_6) (m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6) (m_3 m_4 m_5 m_6 | D_{j_1} (g)) (m_1 m_2 m_3 m_4 m_5 m_6).
\]  

(3.12)
In this equation we substitute \( g' \) for \( g \) and replace each matrix element by the expression
\[
(m | D_j (g') | m') = \sum_{m''} (m | D_j (g) | m'') (m'' | D_j (g') | m')
\] (3.13)

Integrating the equation thus obtained over the group we find, according to (3.12),
\[
\sum_{m''} (-1)^{j-m''} \left( m'' | D_j (g) | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right)
\times\left( m'' | D_j (g') | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right)
\times\left( m'' | D_j (g') | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right)
\] (3.14)

If \( m_1, m_2, \) and \( m_3 \) are so chosen that the common factor in this equation is not zero then, dividing by the Clebsch-Gordan coefficient and dropping one prime everywhere, we obtain the following equation
\[
(-1)^{j-m''} \left( m'' | D_j (g) | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right) =
\sum_{m''} (-1)^{j-m''} \left( m'' | D_j (g) | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right)
\times\left( m'' | D_j (g') | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right)
\times\left( m'' | D_j (g') | m_1 m_2 m_3 | j_1 j_2 j_3 - m'' \right)
\] (3.15)

which will be useful later on.

4. Expressions for the Clebsch-Gordan coefficients and their properties

The Clebsch-Gordan coefficients are frequently encountered in quantum-mechanical calculations. It is therefore useful to have explicit expressions for them in terms of the parameters \( j, j_1, j_2, m, m_1, m_2, m_3, \). Such expressions were obtained in different ways by Wigner /1931/, Van der Waerden /1938/, Racah /1942/ and Majumdar /1936/.

Wigner used an explicit form of the matrices \( D_j \) and obtained the Clebsch-Gordan coefficients from (3.12). Van der Waerden's method consisted of constructing, with the help of spinors, a space in which \( D_j \), \( D_{j_1} \) and \( D_{j_2} \) operate (spinors are quantities which transform according to the representation \( D_j \)). Racah made use of the recursion relations for Clebsch-Gordan coefficients. Finally, Majumdar obtained an equation which related the Clebsch-Gordan coefficients to the hypergeometric series, from which the required expression could be found.

The expressions for the Clebsch-Gordan coefficients obtained by the four methods described above are equivalent, although they differ in their outward form.

Each possesses its own advantages in particular cases. We shall therefore cite the corresponding formulas obtained by all four methods.

Wigner's formula:
\[
\begin{align*}
\langle j_1 j_2 m_1 m_2 | j_1 j_2 j | m_1 m_2 m_3 \rangle &= \delta (m_1 + m_2, m_3) \cdot \Delta (j_1 j_2 j) \\
&\times \left( \frac{(j + m_1)(j - m_1)(j + m_2)(j - m_2)(j + m_3)(j - m_3)}{(2j + 1)(2j + 1)} \right) \frac{1}{2} \\
&\times \sum_{e} (-1)^{j+m+e} z^{(j + j_1 + j_2 - 2j)(j_1 - m_1 - 2j)(j_2 - m_2 - 2j)}/(j_1 + j_2 + j_3 - 2j)
\end{align*}
\] (4.1)

Van der Waerden's formula:
\[
\begin{align*}
\langle j_1 j_2 m_1 m_2 | j_1 j_2 j | m_1 m_2 m_3 \rangle &= \delta (m_1 + m_2, m_3) \cdot \Delta (j_1 j_2 j) \\
&\times \left( \frac{(j + m_1)(j - m_1)(j + m_2)(j - m_2)(j + m_3)(j - m_3)}{(2j + 1)(2j + 1)} \right) \frac{1}{2} \\
&\times \sum_{e} (-1)^{j+m+e} z^{(j + j_1 + j_2 - 2j)(j_1 - m_1 - 2j)(j_2 - m_2 - 2j)}/(j_1 + j_2 + j_3 - 2j)
\end{align*}
\] (4.2)

Racah's formula:
\[
\begin{align*}
\langle j_1 j_2 m_1 m_2 | j_1 j_2 j | m_1 m_2 m_3 \rangle &= \delta (m_1 + m_2, m_3) \cdot \Delta (j_1 j_2 j) \\
&\times \left( \frac{(j - m_1)(j + m_1)(j - m_2)(j + m_2)(j - m_3)(j + m_3)}{(2j + 1)(2j + 1)} \right) \frac{1}{2} \\
&\times \sum_{e} (-1)^{j+m+e} z^{(j + j_1 + j_2 - 2j)(j_1 - m_1 - 2j)(j_2 - m_2 - 2j)}/(j_1 + j_2 + j_3 - 2j)
\end{align*}
\] (4.3)

Majumdar's formula:
\[
\begin{align*}
\langle j_1 j_2 m_1 m_2 | j_1 j_2 j | m_1 m_2 m_3 \rangle &= \delta (m_1 + m_2, m_3) \cdot \Delta (j_1 j_2 j) \\
&\times \left( \frac{(j + m_1)(j - m_1)(j + m_2)(j - m_2)(j + m_3)(j - m_3)}{(2j + 1)(2j + 1)} \right) \frac{1}{2} \\
&\times \sum_{e} (-1)^{j+m+e} z^{(j + j_1 + j_2 - 2j)(j_1 - m_1 - 2j)(j_2 - m_2 - 2j)}/(j_1 + j_2 + j_3 - 2j)
\end{align*}
\] (4.4)

In these formulas
\[
\Delta (j_1 j_2 j) = \left( \frac{(j_1 + j_2 - m) + (2j + 1)}{(j_1 + j_2 + j + 1)} \right)
\]
(4.5)
is the so-called triangle coefficient. The sums in the formulas (4.1) to (4.4) are essentially hypergeometric series, owing to which the Clebsch-Gordan coefficients
can be expressed in terms of the hypergeometric function /Rose 1955/

\( (j_1 j_2 m_1 j_1 j_2 m_2) = 0 \) if

\[
\times \left[ \frac{(j_1 + j_2 - j_1 j_2)(j_1 + j_2 + 1)(j_1 - m_1)(j_2 - m_2)}{(j_1 + j_2 - m_1)(j_1 + j_2 + 1)(j_1 - m_1)(j_2 - m_2)} \right]^{\frac{1}{2}} \times \\
\times \frac{(j_1 + j_2 + m_1)}{(j_1 - j_2 - m_1)} \cdot 2F_1 \left( -j_1 - j_2, j_1 + j_2 - m_1 - 1, -j_1 + j_2 - m_1 - 1 \right).
\]

(4.6)

Using the explicit expressions of the Clebsch-Gordan coefficients, it is possible to obtain the following symmetries /Racah 1942:*

\[
(j_1 j_2 m_1 m_2 j_1 j_2 j_3 j_4) =
\]

\[
= (-1)^{j_1 - j_2 - m_1} \left[ \begin{array}{c} 2j_1 + 1 \\ j_1 + j_2 + 1 \\ -j_1 - j_2 - m_1 \\ j_1 + m_1 \\ j_2 - m_2 \end{array} \right]^{\frac{1}{2}} (j_1 j_2 m_1 - m_2 j_1 j_2 - m_2).
\]

(4.7)

\[
= (-1)^{j_1 - m_1} \left[ \begin{array}{c} 2j_1 + 1 \\ j_1 + j_2 + 1 \\ -j_1 - j_2 - m_1 \\ j_1 + m_1 \\ j_2 - m_2 \end{array} \right]^{\frac{1}{2}} (j_1 m_1 - m_2 j_1 j_2 - m_2).
\]

(4.8)

Combining (4.7) and (4.8) we obtain all the symmetry properties of the Clebsch-Gordan coefficients. It is convenient to transform the sums (4.1)-(4.4) with the aid of the symmetry relations so as to be left with the least number of terms in them.

It should be noted that use of the symmetry properties and replacement of the summation parameter does not transform one expression for the Clebsch-Gordan coefficients into a different one. Transforming from one form into another requires rather complicated algebraic manipulations, such as were carried out, for instance, by Racah 1942/ to transform (4.3) into (4.5).

In certain cases the sums in (4.1)-(4.4) can be reduced to a single term. This happens when the triangle \((j_1, j_2, j_3)\) is "inverted" i.e., for \(j_1 + j_2 = j_3\) or when one of the parameters \(j_1, j_2, j_3\) is zero. It is not difficult to verify that

\[
(j_1 m_1 m_2 j_1 j_2 j_3 j_4) = 0 \) if \(j_1 = m_1, m_2\),
\]

\[
(j_1 j_2 m_1 m_2 j_1 j_2 j_3 j_4) = 0 \) if \(j_1 = m_1, m_2\) \times
\]

\[
\times (-1)^{j_1 - m_1} \left( \begin{array}{c} 2j_1 + 1 \\ j_1 + j_2 + 1 \\ -j_1 - j_2 - m_1 \\ j_1 + m_1 \\ j_2 - m_2 \end{array} \right)^{\frac{1}{2}},
\]

(4.9a)

\[
(x_1 j_2 m_1 m_2 j_1 j_2 j_3 j_4) = 0 \) if \(j_1 = m_1, m_2\) \times
\]

\[
\times (-1)^{j_1 - m_1} \left( \begin{array}{c} 2j_1 + 1 \\ j_1 + j_2 + 1 \\ -j_1 - j_2 - m_1 \\ j_1 + m_1 \\ j_2 - m_2 \end{array} \right)^{\frac{1}{2}},
\]

(4.9b)
The convenience of these formulas consists of the fact that the number of terms in the sums is small and that after expressing \( j \) in terms of \( j_1 \), no factorials depend on \( j_1 \) and \( m \) remain in the denominator under the summation sign.

The algebraic expressions for the Clebsch-Gordan coefficients for the cases \( j_1 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \) and 2 are given in Condon and Shortley's book /1949/. For cases where further \( j_1 = \frac{5}{2}, \frac{7}{2} \) and 3 they are given in Sears and Radtke /1954/. For \( j_1 = \frac{7}{2} \), the expressions were obtained by the authors; for \( j_1 = 3 \) they were taken from Falkoff et al. /1952/. The expressions for \( j_1 = \frac{7}{2} \) were also obtained by Salto and Morita /1955/ and Melvin and Swamy /1957/, and for \( j_1 = 3 \) by Yamada and Morita /1952/. For \( j_1 = \frac{7}{2} \), analogous expressions were obtained by Zakusknas and Mauz /1957/. In all these cases, the Clebsch-Gordan coefficients are expressed in terms of \( j_1 \) and \( m \) for given \( j_2 \) and \( m_2 \), except in Sears and Radtke (loc. cit.) where they have been expressed in terms of \( j \) and \( m \) for given values of \( j_1 \) and \( m_1 \).

An examination of the above-mentioned formulas shows that we can write the Clebsch-Gordan coefficients in the following form

\[
(j_1j_2m - m_2 | j_1j_2j_3 + j) = A_{j_1j_2}(j_1j_2j_3 + j)
\]

Here

\[
A_{j_1j_2}(j_1j_2j_3 + j) = \prod_{x=0}^{j_1} \frac{(2j_1 + 2j_1 - x - 1)}{2x}
\]

where \( k \) varies from \( -j_2 \) to \( +j_2 \). The quantity \( B \) satisfies the following symmetry condition.

\[
B_{j_1j_2j_3 + j}(m) = (-1)^{j_1} B_{j_1j_2j_3 + j}(m).
\]

This property makes it possible to reduce the size of the algebraic tables by nearly one-half, however, it was not used in the compilation of the tables listed above. Owing to the availability of the general expression (4.17) for \( A \), it is possible to restrict oneself to giving the expressions for \( B \) alone (say, for \( m_2 \geq 0 \)). The corresponding tables for existing cases \( j_1 \leq 4 \) are given at the end of the book (appendix 2). Asymptotic expressions for the Clebsch-Gordan coefficients for large values of the parameters may be found in Bussaud and Tolhoek /1957/.

5. Wigner coefficients and their properties

The symmetry properties of the Clebsch-Gordan coefficients assume a far more convenient form if we introduce the Wigner coefficients (Wigner 1937/)

\[
(j_1j_2j_3)
\]

which are related to the Clebsch-Gordan coefficients by the following formulas:

\[
(j_1j_2j_3 | j_1j_2m_2) = (-1)^{j_1} B_{j_1j_2j_3 + j}(m_2).
\]
Here and hereafter (f) will denote 2f + 1.

In accordance with the conditions of nonvanishing of the Clebsh–Gordan coefficients (5.8), the Wigner coefficients are nonzero when the parameters of the upper row \( j_1, j_2, j_3 \) form a triad, i.e., can be the sides of a triangle with an integral perimeter, and when the sum of the parameters \( m_1, m_2, m_3 \) of the lower row is zero. The parameters \( j_1, j_2, j_3, m_1, m_2, m_3 \) can be integral or half-integral; further, \( j_i \) and \( m_i \) are simultaneously integral or half-integral, so that each of the nine numbers

\[
\begin{align*}
  j_1 + m_1, & \quad j_2 + m_2, & \quad j_3 + m_3, \\
  j_1 - m_1, & \quad j_2 - m_2, & \quad j_3 - m_3, \\
  j_1 + j_2 + j_3,
\end{align*}
\]

is an integer.

From (4.7) one can easily obtain the following permutation properties of the Wigner coefficients

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \varepsilon (j_1 \ j_2 \ j_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_2 & m_3 & m_1 \end{pmatrix},
\]

(5.4)

where \( \varepsilon = 1 \) if the permutation \( 1 \ 2 \ 3 \) is even and \( \varepsilon = (-1)^{j_1+j_2+j_3} \) if it is odd.

From (4.8) it follows that

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_2 & m_3 & m_1 \end{pmatrix},
\]

(5.5)

The orthogonality properties of the Clebsh–Gordan coefficients (3.7) give the following orthogonality properties of the Wigner coefficients

\[
\sum_{j_1, j_2, j_3} (-1)^{j_1+m_1-j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(j_1, j_2) (j_3^{-1} \delta(m_1, m_2) (-1)^j \delta(j_1 j_2 j_3)
\]

(5.6a)

and

\[
\sum_{j_1, j_2, j_3} (-1)^{j_1-m_1-j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-m_1} \delta(m_1, m_2) (-1)^{j_1-m_1} \delta(m_1, m_2).
\]

(5.6b)

\( \{j_1 j_2 j_3\} \) is a triangular delta which is equal to unity if \( j_1, j_2, j_3 \) form a triad and zero otherwise. The forms of (5.6a) and (5.6b) were chosen for convenience in the subsequent considerations.

From (4.9b) it follows that

\[
\begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix},
\]

(5.7)

Using (5.2) one can write explicit expressions and recursion formulas for the Wigner coefficients. We note that by using the symmetries one can reduce the sum of the form (4.1)–(4.4) to a form in which the number of terms in the sum over \( z \) is only larger by one than the smallest of the nine numbers in (5.3).

In the frequently-encountered particular case \( m_1 = m_2 = m_3 = 0 \), it follows from (4.11) that

\[
\begin{pmatrix} j_1 & j_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \gamma \end{pmatrix} (\text{even} \quad j_1 + j_2 + j_3 = 2g)
\]

(5.8a)

and

\[
\begin{pmatrix} j_1 & j_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 (\text{odd} \quad j_1 + j_2 + j_3 = 2g)
\]

(5.8b)

for even \( j_1 + j_2 + j_3 = 2g \) and odd \( j_1 + j_2 + j_3 = 2g \). We note that the quantity \( \{j_1 j_2 j_3\} \) is fully symmetric in all parameters.

We give yet another property of the Wigner coefficients which was obtained by Regge /1958/. For convenience an intermediate notation is introduced

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 + j_2 + j_3, & j_3 - j_2 - j_1, & j_2 - j_3 - j_1 \\ j_1 - m_1, & j_2 - m_2, & j_3 - m_3 \\ j_1 + m_1, & j_2 + m_2, & j_3 + m_3 \end{pmatrix}
\]

(5.9)

According to (5.3) all the nine numbers in the right-hand side are positive integers. The sum of the numbers in each row and column is \( j_1 + j_2 + j_3 \). In this notation one can arbitrarily permute the rows or columns, as well as transpose them relative to the principal diagonal. An even permutation of the rows and columns and transposition does not change the Wigner coefficient. Upon an odd permutation of the rows or columns the Wigner coefficient is multiplied by \((-1)^{j_1+j_2+j_3}\). The permutation of columns is equivalent to (5.4), and the permutation of the two lower rows to (5.5).

The other permutations and transposition give essentially new symmetries.

Replacing the Clebsh–Gordan coefficients in (3.15) by the Wigner coefficients, we obtain the following important relation

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sum_{j_1, j_2, j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} m_1 | D_1(g) | m_2 \end{pmatrix} \begin{pmatrix} m_2 | D_2(g) | m_3 \end{pmatrix} \times
\]

\[
\times \begin{pmatrix} m_3 | D_3(g) | m_3 \end{pmatrix}
\]

(5.10)

Instead of the Wigner coefficients, certain authors use other quantities related to them (see Appendix 1). All of these are less symmetrical than the Wigner coefficients, and are therefore much less convenient to use.
Chapter II

ADDITION OF AN ARBITRARY NUMBER OF ANGULAR MOMENTA

The addition of two angular momenta, examined in the preceding chapter, may be immediately generalized to the addition of an arbitrary number of angular momenta. The present chapter is devoted to the study of this problem. In section 6 we examine the general questions concerning the addition of angular momenta and introduce the generalized Clebsch-Gordan coefficients. In the following section we give the connection between the reduction of the direct product of an arbitrary number of representations of the rotation group and the construction of wave functions of the coupled angular momenta.

Upon passing from two to a greater number of angular momenta, the resultant angular momentum and magnetic quantum numbers are no longer sufficient for a complete characterization of the wave function owing to the appearance of intermediate angular momenta. Furthermore, the wave functions of the coupled angular momenta depend upon the scheme of addition. The matrices of transformations between wave functions belonging to different schemes of addition of the angular momenta are discussed in section 8. The next section deals with the problem of simplifying these matrices.

The last section of this chapter introduces a more symmetric quantity than the generalized Clebsch-Gordan coefficient, in the same manner as was done in section 5 for the case of two angular momenta.

6. General considerations on the addition of an arbitrary number of angular momenta

Let us first consider three angular momentum operators \( J_1, J_2 \) and \( J_3 \). We shall denote an eigenfunction of the commuting set of independent operators

\[ \{J_1, J_2, J_3, J_{12}, J_{13}, J_{23}, J_{123} \} \]

by

\[ \psi(\{j_1, j_2, j_3, m_1, m_2, m_3\}) = \psi(\{j_1, m_1\}) \psi(\{j_2, m_2\}) \psi(\{j_3, m_3\}). \]

(6.1)

We may consider a different set of commuting operators, such as

\[ \{J_1, J_2, J_3, J_{12}, J_{23}, J_{13}, J_{123} \} \]

(6.2)

Here

\[ J_{12} = J_1 + J_2 \]

(6.3)

and

\[ J = J_1 + J_2 + J_3 \]

(6.4)

We shall call their eigenfunction \( \psi(\{j_1, j_2, j_3, j_{12}, j_{13}, j_{23}, j_{123}\}) \) the wave function of the coupled angular momenta. It may be expressed in terms of the functions (6.2) as follows:

\[ \psi(\{j_1, j_2, j_3, j_{12}, j_{13}, j_{23}, j_{123}\}) = \sum_{\substack{m_1, m_2, m_3, j_{12}, j_{13}, j_{23}, j_{123}}} \psi(\{j_1, m_1\}) \psi(\{j_2, m_2\}) \psi(\{j_3, m_3\}) \times \psi(\{j_{12}, m_1, m_2, j_{13}, j_{23}, j_{123}\}) \psi(\{j_{23}, m_3, j_{12}, j_{13}, j_{123}\}) \psi(\{j_{13}, m_2, m_1, j_{12}, j_{23}, j_{123}\}) \psi(\{j_{123}, m_1, m_2, m_3\}). \]

(6.5)

The quantities

\[ \{j_{12}, j_{13}, j_{23}, j_{123}\} \]

(6.6)

which enter into the above expression shall be called the generalized Clebsch-Gordan coefficients for three coupled angular momenta [Levinson 1957a]. The quantum number \( j_{123} \) of the operator \( J_{123} \) will be called the intermediate angular momentum parameter, or simply the intermediate angular momentum.

Functions of coupled angular momenta can be constructed from the formulas of vector addition of two angular momenta. Using (3.3) twice we have

\[ \psi(\{j_1, j_2, j_{13}, j_{12}, j_{123}\}) = \sum_{\substack{m_1, m_2, m_3, j_{12}, j_{13}, j_{123}}} \psi(\{j_1, m_1\}) \psi(\{j_2, m_2\}) \psi(\{j_{13}, m_3\}) \times \psi(\{j_{12}, m_1, m_2, j_{123}\}) \psi(\{j_{13}, m_3, j_{12}, j_{123}\}) \psi(\{j_{123}, m_1, m_2, j_{13}, j_{12}, j_{123}\}). \]

(6.7)

Comparing (6.5) and (6.7) we obtain the following expression for the generalized Clebsch-Gordan coefficient:

\[ \{j_{12}, j_{13}, j_{23}, j_{123}\} = \sum_{\substack{m_1, m_2, m_3, j_{12}, j_{13}, j_{123}}} \psi(\{j_1, m_1\}) \psi(\{j_2, m_2\}) \psi(\{j_{13}, m_3\}) \times \psi(\{j_{12}, m_1, m_2, j_{123}\}) \psi(\{j_{13}, m_3, j_{12}, j_{123}\}) \psi(\{j_{123}, m_1, m_2, m_3\}). \]

(6.8)

The summation above is purely formal, as it follows from the condition of nonvanishing of the Clebsch-Gordan coefficients that the parameter of summation \( M_{123} \) can have only the value \( m_1 + m_2 + m_3 \).

If the subscripts of the intermediate angular momentum denote the angular momenta from which it is compounded, the set of subscripts of all intermediate and resultant angular momenta indicates the scheme of coupling of the angular momenta. Thus, the function

\[ \psi(\{j_{12}, j_{13}, j_{23}, j_{123}\}) \]

(6.9)

is constructed by the coupling \( J_1 + J_2 = J_{12}, J_3 + J_4 = J_{13} \) and \( J_{23} + J_{12} = J_{123} \), which we shall abbreviate as

\[ A = \{(1 + 3) + (2 + 4)\}. \]

(6.10)

Here and in the following the capital letters \( A, B, \ldots \) will denote the coupling scheme. In the coupling scheme it is convenient to differentiate between the "distribution" of intermediate couplings, determined by the arrangement of the brackets in the notation (6.10), and the "coupling sequence", determined by the arrangement of the indices of the component angular momenta \( j_i \) in the same [Levinson and Vanagas 1957]. For the wave functions of coupled angular momenta it is convenient to use a notation which sets the coupling scheme in evidence. Thus, instead of (6.9)
we write
\[ \psi \left( \left( j_1, i_1, j_{1s}, j_{1s}, j_{2}, j_{2s}, j_{2s}, \ldots, j_{ns}, j_{ns} \right) \right) \] (6.11)

In this notation there is no loss of clarity even when the indices are suppressed. Where a concrete specification of the coupling scheme is not required, one can write, instead of (6.11)
\[ \psi \left( j_1, i_1, j_2, j_2, \ldots, j_n, j_n \right) \] (6.12)
where \( a_1 \) and \( a_2 \) are the intermediate angular momenta. In the general case a wave function of the coupled angular momenta is written in a similar fashion
\[ \psi \left( j_1 \cdots j_n a \right) \] (6.13)
where \( a \) is the set of intermediate angular momenta \( a_1, a_2, \ldots, a_n \). It is the eigenfunction of the set of operators
\[ \mathbf{J}_1, \mathbf{J}_2, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{J}_n \] (6.14)
In terms of the eigenfunctions of the operators
\[ \mathbf{J}_1, \mathbf{J}_2, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \mathbf{J}_n \] (6.15)
(6.13) is expressed, like (6.4), as
\[ \psi \left( j_1, i_1, j_2, j_2, \ldots, j_n, j_n \right) = \sum m_j \psi \left( j_1, m_j \right) \psi \left( i_1, m_j \right) \psi \left( j_2, m_j \right) \cdots \psi \left( j_n, m_j \right) \] (6.16)
The coefficients of this transformation are the generalized Clebsch-Gordan coefficients for an arbitrary number of angular momenta and an arbitrary scheme of coupling. In the notation (6.16) for the generalized Clebsch-Gordan coefficient it is natural to write the usual Clebsch-Gordan coefficient as \( \left( j_1, m_1, j_2, m_2 \right) \) \( \left( j_1, j_2 \right) \) \( m_1 \), as in Fano (1951).

It is often convenient to specify the coupling sequence in terms of the permutation
\[ P = \begin{pmatrix}
1 & 2 & \cdots & n \\
1 & 2 & \cdots & n
\end{pmatrix} \] (6.17)
which must be carried out on the natural sequence \( 1, 2, \ldots, n \) to obtain the required sequence. For given \( P \) one should understand by \( A \) only the distribution, and drop the indices in the writing of \( A \). Then (6.10) will be represented by
\[ P = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4
\end{pmatrix} \] (6.10a)
\[ A = \left( ( \cdot \cdot \cdot ) ( \cdot \cdot \cdot ) \right) \] (6.10b)
where the dots indicate the positions in which the angular momenta are arranged according to the lower row of (6.10a).

For the above characterization of the scheme of addition the eigenfunctions of

the coupled angular momenta will be written as follows:
\[ \psi \left( j_1, i_1, \ldots, j_n a \right) \] (6.18)

It should also be noted that in this and analogous formulas the indices indicate not only the serial numbers of the angular momenta but also the corresponding coordinates. This means that the permutation operator (6.17) also acts on the coordinates. For this reason (6.18), in an amplified form, will appear as
\[ \psi \left( j_1, i_1, \ldots, j_n a \right) \] (6.19)
where \( i_1, \ldots, i_n \) indicate the coordinates with the same indices.

The indices of the coordinates must be explicitly written when the coupled angular momenta are numerically equal, such as in
\[ \psi \left( \left( \cdots \right) \left( \cdots \right) \right) \] (6.20)

7. Group-theoretic considerations on the generalized Clebsch-Gordan coefficients

From section 3 we know that the usual Clebsch-Gordan coefficients are the elements of the matrix which reduces the direct product of two irreducible representations of the three-dimensional rotation group. It is not difficult to see that the generalized Clebsch-Gordan coefficients are elements of the matrix which reduces the direct product of an arbitrary number of irreducible representations of the same group. We shall satisfy ourselves of this by an explicit examination of the direct product.

Let us suppose that we are given the direct product of \( n \) irreducible representations
\[ \mathbf{D}_i \times \mathbf{D}_j \times \cdots \times \mathbf{D}_k \] (7.1)
The matrix which reduces it to the 'diagonal' form shall be denoted by \( \mathbf{G}_{ij} \). We then have
\[ \mathbf{G}_{ij} \mathbf{G}_{ij}^{-1} \] (7.2)
where \( \mathbf{G}_{ij} \) denotes the number of submatrices with the same value of \( j_1 \) further
\[ a_j + \cdots + a_n = 1. \] (7.3)
Clearly, the following relation holds true

\[ \sum_j a_j (2j+1) = \prod_{i=1}^n (2l_i + 1), \tag{7.4} \]

as the order of the decomposed matrix should equal the order of the one from which we had started.

The reduction indicated in (7.2) is carried out by the successive reduction of the direct products of two irreducible representations. It is made definite by specifying the scheme of reduction, which is the group-theoretic analogue of the scheme of addition of the preceding section. It is therefore denoted by the symbols of the preceding section. Thus, if the scheme of reduction is such that first the direct product \( D_{i_1} \times D_{i_2} \) is reduced and then \( D_{j_1} \times D_{j_2} \), the order is symbolised by the formula \( A' = (1+2)+3 \).

Each reduction of the product of two representations results in the appearance of a usual Clebsch-Gordan coefficient. Owing to this the elements of the matrix \( C_{i_1}^{j_1, j_2} \) are expressed in terms of the elements of the matrix \( C \) with two subscripts. In other words, the generalized Clebsch-Gordan coefficients are expressed in terms of the usual ones, as we already saw in the preceding section. For example, let \( n = 3 \) in (7.2) and the scheme of reduction be \( A' \); we then have

\[ C_{j_1, j_2}^{i_1} (D_{i_1} \times D_{i_2} \times D_{j_1}) \cdot C_{j_3}^{j_2} = C_{j_1, j_3}^{i_1} \cdot \left( C_{j_2}^{j_1} (D_{i_1} \times D_{i_2}) \cdot C_{j_3}^{j_2} \cdot D_{j_1} \right) \cdot C_{j_1}^{j_3} \cdot i_2. \tag{7.5} \]

Using (3.11a), the right-hand side of the above may be expressed in the following form

\[ \sum_{i_{123}} (J_{123}JM | J_{123}JM) (j_{12}M_3 | j_{12}M_3) (j_{13}M_2 | j_{13}M_2) (j_{23}M_1 | j_{23}M_1) \times \]

\[ \times (m_1m_1m_1 | D_{i_1} \times D_{i_2} \times D_{j_1}) \cdot m_1m_1m_1 \times \]

\[ \times (j_{12}m_2M_2 | j_{12}M_2j_{13}M_1) (j_{13}M_2j_{23}M_1 | J_{13}JM), \tag{7.6} \]

where the summation is carried out over \( m_1, m_2, m_3 \). A comparison with (6.8) shows that the elements of the matrix \( C_{i_1}^{j_1, j_2} \) are the generalized Clebsch-Gordan coefficients for the case of three angular momenta and the scheme of reduction \( A' \).

The matrix which reduces the direct product (7.1) is unitary. Consequently, we have the following conditions of orthornormality of the generalized Clebsch-Gordan coefficients

\[ \sum_{m_{123}} (j_{12}m_2j_{23}M_1 | j_{12}m_2j_{23}M_1) \times \]

\[ \times (j_{12}m_2 | j_{12}m_2) (j_{12}m_2 | j_{12}m_2) = \delta (s, s') \delta (J, J') \delta (M, M'), \tag{7.7a} \]

and

\[ \sum_{m_{123}} (j_{12}m_2j_{23}M_1 | j_{12}m_2j_{23}M_1) \times \]

\[ \times (j_{12}m_2 | j_{12}m_2) (j_{12}m_2 | j_{12}m_2) = \delta (s, s') \delta (J, J') \delta (M, M'), \tag{7.7b} \]

where

\[ \delta (a, a') = \delta (a_1, a_1) - \delta (a_2, a_2) - \delta (a_3, a_3). \tag{7.7c} \]

Further, the generalized Clebsch-Gordan coefficients satisfy the reality condition

\[ (j_{12}m_2j_{23}M_1 | j_{12}m_2j_{23}M_1) = \]

\[ = (j_{12}m_2j_{23}M_1 | j_{12}m_2j_{23}M_1), \tag{7.8} \]

which is the direct consequence and generalization of (3.6).

It is easily seen that the generalized Clebsch-Gordan coefficients are nonzero when

\[ \sum_{i_{123}} J_{123}JM \geq 2M - \sum_{i_{123}} J_{123}JM, \tag{7.9a} \]

\[ \sum_{i_{123}} J_{123}JM \geq \text{integer}, \tag{7.9b} \]

\[ \sum_{i_{123}} J_{123}JM \geq M, \tag{7.9c} \]

which are the generalizations of (3.8). They may be referred to as the polygon conditions.

We also give the following formula

\[ (m_1 | D_{i_1}m_1) \times (m_1 | D_{i_1}m_1) = \]

\[ = \sum_{a_1 \neq \Phi} (j_{12}m_2j_{23}M_1 | (j_{12}m_2j_{23}M_1) \times \]

\[ \times (j_{12}m_2 | j_{12}m_2) (j_{12}m_2 | j_{12}m_2) = \delta (s, s') \delta (J, J') \delta (M, M'), \tag{7.10} \]

which is the direct generalization of (3.11b).

8. The transformation matrix

Let us examine two systems of wave functions of coupled angular momenta constructed by the two different schemes of addition \( A \) and \( B \). We shall represent the passage from one system into the other in the following form

\[ \psi (j_{12}m_2j_{23}M_1) = \]

\[ = \sum_{a_1 \neq \Phi} (j_{12}m_2j_{23}M_1) \times \]

\[ \times (j_{12}m_2 | j_{12}m_2) (j_{12}m_2 | j_{12}m_2) = \delta (s, s') \delta (J, J') \delta (M, M'), \tag{7.11} \]

where

\[ \delta (s, s') = \delta (s_1, s_1) - \delta (s_2, s_2) - \delta (s_3, s_3). \tag{7.12} \]

Further, the generalized Clebsch-Gordan coefficients satisfy the reality condition

\[ (j_{12}m_2j_{23}M_1 | j_{12}m_2j_{23}M_1) = \]

\[ = (j_{12}m_2j_{23}M_1 | j_{12}m_2j_{23}M_1), \tag{7.13} \]

which is the direct consequence and generalization of (3.6).

It is easily seen that the generalized Clebsch-Gordan coefficients are nonzero when

\[ \sum_{i_{123}} J_{123}JM \geq 2M - \sum_{i_{123}} J_{123}JM, \tag{7.14} \]

\[ \sum_{i_{123}} J_{123}JM \geq \text{integer}, \tag{7.15} \]

\[ \sum_{i_{123}} J_{123}JM \geq M, \tag{7.16} \]

which are the generalizations of (3.8). They may be referred to as the polygon conditions.

We also give the following formula

\[ (m_1 | D_{i_1}m_1) \times (m_1 | D_{i_1}m_1) = \]

\[ = \sum_{a_1 \neq \Phi} (j_{12}m_2j_{23}M_1 | (j_{12}m_2j_{23}M_1) \times \]

\[ \times (j_{12}m_2 | j_{12}m_2) (j_{12}m_2 | j_{12}m_2) = \delta (s, s') \delta (J, J') \delta (M, M'), \tag{7.17} \]

which is the direct generalization of (3.11b).

8. The transformation matrix

Let us examine two systems of wave functions of coupled angular momenta constructed by the two different schemes of addition \( A \) and \( B \). We shall represent the passage from one system into the other in the following form

\[ \psi (j_{12}m_2j_{23}M_1) = \]

\[ = \sum_{a_1 \neq \Phi} (j_{12}m_2j_{23}M_1) \times \]

\[ \times (j_{12}m_2 | j_{12}m_2) (j_{12}m_2 | j_{12}m_2) = \delta (s, s') \delta (J, J') \delta (M, M'), \tag{7.18} \]
where the second factor in the right-hand side is the matrix which transforms the eigenvectors of the scheme of addition \( A \) to the eigenvectors of the scheme \( B \). For the sake of brevity, from now on we shall refer to this matrix as the transformation matrix.

Multiplying (8.1) by \( \psi^* \) and integrating over all the variables, we obtain the equation

\[
\langle (j_1 \cdots j_o)^a a J'M' | (j_1 \cdots j_o)^a b J'M \rangle = \varepsilon(j_1, j_2) \varepsilon(M, M') \int \psi^* (j_1 \cdots j_o)^a a J'M' \psi (j_1 \cdots j_o)^a b J'M' dx.
\]

Taking this into account we conclude that

\[
\langle (j_1 \cdots j_o)^a a J'(j_1 \cdots j_o)^b b J' \rangle = \varepsilon(j_1 \cdots j_o)^a b J'(j_1 \cdots j_o)^b a J',
\]

i.e., the transformation matrix is real if the Clebsch-Gordan coefficients are real. Remembering that the transformation matrix is independent of \( M \), one can write (8.4) as follows:

\[
\langle (j_1 \cdots j_o)^a a J'(j_1 \cdots j_o)^b b J' \rangle = \varepsilon(j_1 \cdots j_o)^a b J'(j_1 \cdots j_o)^b a J'.
\]

For the particular case \( A = B \) and \( j_1 = j_2 = \cdots = j_o \) we have, according to (8.7),

\[
\langle (j_1 \cdots j_o)^a a J'(j_1 \cdots j_o)^b b J' \rangle = \varepsilon(j_1 \cdots j_o)^a b J'(j_1 \cdots j_o)^b a J'.
\]

Further, using (8.4) we obtain

\[
\langle (j_1 \cdots j_o)^a a J'(j_1 \cdots j_o)^b b J' \rangle = \varepsilon(j_1 \cdots j_o)^a b J'(j_1 \cdots j_o)^b a J'.
\]

In view of the fact that

\[
P_i P_i^{-1} = (P_i P_i^{-1})^{-1} = P
\]

it follows from (8.10) that the two such matrices given by the permutations \( P \) and \( P^{-1} \) are equal.

If in the left- and right-hand sides of the matrix there are intermediate angular moments which differ only in the order of the subscripts, such as \( J_{1\ell}^\ell \) and \( J_{\ell 1} \), then the matrix is diagonal in these angular moments. This follows from the fact that the two intermediate angular moments are the eigenvalues of the same operator \((J_1 + J_2 + J_3)^a \). The order of indices of these angular moments is therefore immaterial.

We note that upon permutation of two directly coupled angular moments (these may be either the coupled angular momenta \( J_k \) or the intermediate angular momenta), the transformation matrix is multiplied by a phase factor. Thus

\[
\langle (j_1 j_2 J_{1\ell} j_3 j_4 J_{2\ell} j_5 j_6 J_{3\ell}) | \langle (j_1 j_2 J_{1\ell} j_3 j_4 J_{2\ell} j_5 j_6 J_{3\ell}) \rangle = \varepsilon(-j_1 \epsilon_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9, j_{10}, j_{11}, j_{12}) \times
\]

\[
\varepsilon(j_1 j_2 J_{1\ell} j_3 j_4 J_{2\ell} j_5 j_6 J_{3\ell}) J | \langle (j_1 j_2 J_{1\ell} j_3 j_4 J_{2\ell} j_5 j_6 J_{3\ell}) \rangle.
\]
Here in the left-hand side of the matrix $J_{Lq}$ and $J_{Bq}$, as well as $j_1$ and $j_2$, have been permuted. Upon permutation of the angular momenta $J_{Lq}$ and $J_{Bq}$ there results, in accordance with (4,7c), the additional phase factor $(-i)^{q+n}J_{q-n}$.

9. Simplification of the transformation matrix

Let us consider the following transformation matrix

$$\left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f\right)\right).$$  \hspace{1cm} (9.1)

Here on the left-hand side the angular momenta $l_1, \ldots, l_p$ are added up to $L$ according to the scheme $A$, $B$ is then added to $j_1, \ldots, j_q$ according to scheme $B$. On the right-hand side we have, instead of the schemes $A$ and $B$, the schemes $\tilde{A}$ and $\tilde{B}$ respectively. This matrix, as follows from the end of the last section, is diagonal in $L$. We can regard the transformation of a scheme of coupling of the set $l_1, \ldots, l_p$ as a transformation of the eigenfunctions of the operator $L^\rho$, which does not depend on the scheme of coupling of the angular momenta $l_1, \ldots, l_p$. Consequently, in accordance with (3.4), (9.1) can be factorized into

$$\delta(L, L') \delta \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.2)

The first factor corresponds to the transformation of the set of angular momenta $l_1, \ldots, l_p$, and the second factor to the transformation of the coupling of the resultant angular momenta of this set to other angular momenta. If one of these transformations is the identity, the corresponding matrix element degenerates into a product of Kronecker deltas. If the set $l_1, \ldots, l_p$ is transformed identically, i.e., $\tilde{A} = A$, then

$$\delta(L, L') \delta \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.3)

If the scheme of coupling of the angular momentum $L$ to the angular momenta $l_1, \ldots, l_p$ is transformed identically, i.e., $B = B'$, then

$$\delta(L, L') \delta \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.4)

Let us consider, for instance, the matrix

$$\left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.5)

This matrix produces a transformation which consists of changing the scheme of coupling within the set $J_{Lq} J_{Bq}$ and of changing the scheme of coupling of the resultant angular momentums of this set to the angular momenta $j_1, j_2, j_3$. On the basis of (5.2) the matrix element (9.5) therefore becomes

$$\delta(L, L') \delta \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.6)

We further note that if one of the component angular momenta is zero, it may be dropped when writing the transformation matrix. Thus, for $j_2 = \mathbf{0}$

$$\delta(L, L') \delta \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.7)

This equality follows from (4.9a).

Let us consider two cases in which the transformation is susceptible of simplification. In the first case the transformation only removes one angular momentum $j_3$ from the set $l_1, \ldots, l_p, j_3$. This matrix may be simplified as follows:

$$\delta(L, L') \delta \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right) \times$$

$$\times \left((l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f) \right).$$  \hspace{1cm} (9.8)

To be convinced of the validity of this equation one need only write the matrix in the following form

$$\left(\ldots \ldots \ldots \right) = \sum (\ldots (P) (P) \ldots \ldots \ldots),$$  \hspace{1cm} (9.9)

with the intermediate coupling of the angular momenta

$$\Pi = (l_1 \ldots l_p \cdot a \cdot L \cdot l_q \cdot b \cdot f).$$  \hspace{1cm} (9.10)
and use (9.4) and (9.3) for the matrices under the summation sign. The summation in (9.9) is carried out over the intermediate angular momenta $e' \mathbf{L} j_{6e'}$.

Let us, for instance, consider the matrix

$$
\left|
\begin{array}{c}
((j_{12}j_{6}j_{5}j_{4}j_{3}j_{2}j_{1}) J_{123456}) J \\
((j_{12}j_{6}j_{5}j_{4}j_{3}j_{2}j_{1}) J_{123456}) J
\end{array}
\right|
$$

(9.11)

The transformation which it produces removes the angular momentum $j_{6}$ from the set $j_{5}, j_{4}, j_{3}$ in the left-hand side and forms the set $j_{4}, j_{3}$ in the right side. From (9.8), the matrix element (9.11) is therefore

$$
\left|
\begin{array}{c}
((j_{12}j_{6}j_{5}j_{4}j_{3}j_{2}j_{1}) J_{123456}) J_{123456}
\end{array}
\right|
\times
\left|
\begin{array}{c}
((j_{12}j_{6}j_{5}j_{4}j_{3}j_{2}j_{1}) J_{123456}) J_{123456}
\end{array}
\right|
$$

(9.12)

The transformation matrix may also be simplified when in its left-hand side the angular momenta $j_{1}, \ldots, j_{6}$ attach themselves to the coupled set of angular momenta $j_{1}, \ldots, j_{6}$, while in the right-hand side, as opposed to it, $j_{1}, \ldots, j_{6}$ attach themselves to the coupled set $j_{1}, \ldots, j_{6}$. The simplification has the following form:

$$
\left|
\begin{array}{c}
((j_{12}j_{6}j_{5}j_{4}j_{3}j_{2}j_{1}) J_{123456}) J_{123456}
\end{array}
\right|
\times
\left|
\begin{array}{c}
((j_{12}j_{6}j_{5}j_{4}j_{3}j_{2}j_{1}) J_{123456}) J_{123456}
\end{array}
\right|
$$

This equation may be established, like (9.8), by choosing an intermediate scheme of the form

$$
\Pi = (j_{1}, \ldots, j_{6})_{a \mathbf{L}(j_{1}, \ldots, j_{6})_{a \mathbf{L}}} J.
$$

(9.14)

Consider, for example, the matrix

$$
\left|
\begin{array}{c}
((\mathbf{L} j_{12} j_{6} j_{5} j_{4} j_{3} j_{2} j_{1}) J_{123456}) J
\end{array}
\right|
\times
\left|
\begin{array}{c}
((\mathbf{L} j_{12} j_{6} j_{5} j_{4} j_{3} j_{2} j_{1}) J_{123456}) J
\end{array}
\right|
$$

(9.15)

In this matrix the angular momenta $j_{6}, j_{5}$, and $j_{6}$ are attached to the set $j_{1}, j_{2}$ in the left-hand side, and $j_{1}, j_{2}$ are attached to the set $j_{6}, j_{5}$ in the right-hand side. From (9.13) this matrix is therefore

$$
\left|
\begin{array}{c}
((\mathbf{L} j_{12} j_{6} j_{5} j_{4} j_{3} j_{2} j_{1}) J_{123456}) J
\end{array}
\right|
\times
\left|
\begin{array}{c}
((\mathbf{L} j_{12} j_{6} j_{5} j_{4} j_{3} j_{2} j_{1}) J_{123456}) J
\end{array}
\right|
$$

(9.16)

The formulas obtained for the simplification of the transformation matrices considerably facilitate their calculation and are widely used in deriving various formulas which contain the transformation matrix or similar quantities.

10. Generalized Wigner coefficients and their properties

Just as the usual Clebsch-Gordan coefficients may be expressed in terms of the Wigner coefficients, the generalized Clebsch-Gordan coefficients may be expressed in terms of the generalized Wigner coefficients. Let us define

$$
\left|
\begin{array}{c}
(i_{1} m_{1} \cdots i_{n} m_{n} j_{1} \cdots j_{m} a_{1} \cdots a_{n-2} j M)
\end{array}
\right| = (-1)^{j_{1}+j_{2}+\cdots+j_{m}} (a_{2} \cdots a_{n-2})^2
$$

(9.17)

$$
\times (-1)^{J-M} (j_{1} \cdots j_{m} a_{1} \cdots a_{n-2})^A.
$$

(10.1)

Here $f_{4}$ is a linear combination of its parameters which depends on the scheme of addition $A$. The last factor in (10.1) is the generalized Wigner coefficient. Setting $f_{4} = f_{4+1}$ and replacing $n + 1$ by $n$, the generalized Wigner coefficient may be represented as follows

$$
\left|
\begin{array}{c}
(j_{1} \cdots j_{m} a_{1} \cdots a_{n-2} j M)
\end{array}
\right| = (j_{1} \cdots j_{m} a_{1} \cdots a_{n-2})^A
$$

(10.2)

The conditions of nonvanishing of the generalized Wigner coefficients follow from the conditions of nonvanishing of the generalized Clebsch-Gordan coefficients (7.9). They are:

1. The parameters $j_{1}, j_{2}, \ldots, j_{m}$ must constitute a polygon with an integral perimeter.
2. The sum of the lower parameters $m_{i}$ must be zero.

In the particular case where $m_{1} = \cdots = m_{n-2} = 0$, it follows from (5.8) that the parameters $j_{1}$ must form a polygon with an even perimeter. This is identical with the condition of nonvanishing of the integral of the product of $n$ associated Legendre polynomials.}

In view of the fact that the generalized Clebsch-Gordan coefficients may be
expressed in terms of the corresponding usual coefficients, the generalized Wigner coefficients may be expressed in terms of the usual Wigner coefficients. The form of this expression depends on the scheme of addition $A$. Thus, for the scheme of addition $A = (1 + 2) + (3 + 4)$ we have

$$
\begin{pmatrix}
 j_1 & j_2 & j_3 & j_4 \\
 m_1 & m_2 & m_3 & m_4
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
 j_1 & j_2 & j_3 & j_4 \\
 m_1 & m_2 & m_3 & m_4
\end{pmatrix}
$$

$$
= \sum_{M_0, M_4} \left( -1 \right)^{j_1 + j_2 + j_3 - j_4}
\begin{pmatrix}
 j_1 & j_2 & j_3 \\
 m_1 & m_2 & m_3 - M_4
\end{pmatrix}
\begin{pmatrix}
 j_4 \\
 m_4 - M_0
\end{pmatrix}
\begin{pmatrix}
 j_1 & j_2 & j_3 & j_4 \\
 m_1 & m_2 & m_3 & m_4
\end{pmatrix}
$$

Further, in the equation which corresponds to (10.1),

$$
f_4 (j_1 j_2 j_3 j_4) = j_1 - j_2 - j_3 + j_4 - J.
$$

The permutational symmetries of the generalized Wigner coefficients result from the corresponding symmetries of the usual Wigner coefficients (5.4). This means that permutation of the columns is possible only within individual Wigner coefficients. The symmetry property related to change in the signs of the projections $m_j$ is of the form

$$
\begin{pmatrix}
 j_1 & \cdots & j_n \\
 m_1, \ldots, m_n
\end{pmatrix}
$$

$$
= \left( -1 \right)^{J + \cdots + J + j_1 - j_n}
\begin{pmatrix}
 j_1 & \cdots & j_n \\
 -m_1, \ldots, -m_n
\end{pmatrix}
$$

To prove this one must apply (5.5) to the usual Wigner coefficients in the expression of the generalized Wigner coefficients in terms of the former, change the signs of the projections of all the intermediate angular momenta, and remember that $\left( -1 \right)^{j_1 + \cdots + j_n}
= \left( -1 \right)^{J + \cdots + J}$.

The generalized Wigner coefficients are more symmetrical than the generalized Clebsch-Gordan coefficients, owing to the fact that the entire series of generalized Clebsch-Gordan coefficients is expressed in terms of a single generalized Wigner coefficient. Thus, the generalized Clebsch-Gordan coefficients for the schemes of addition $(1 + 2) + (3 + 4)$ and $(1 + 2) + (3 + 4)$ are expressed in terms of the same generalized Wigner coefficient:

$$
\begin{pmatrix}
 j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4 \\
 \left( j_1 j_2 j_3 j_4 \right) J_{1234} J_{12}
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
 j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4 \\
 \left( j_1 j_2 j_3 j_4 \right) J_{1234} J_{12}
\end{pmatrix}
$$

$$
= \left( -1 \right)^{-j_1 - j_2 + j_3 + j_4}
\begin{pmatrix}
 j_1 & j_2 & j_3 & j_4 \\
 m_1 & m_2 & m_3 - M_4
\end{pmatrix}
\begin{pmatrix}
 j_1 & j_2 & j_3 & j_4 \\
 m_1 & m_2 & m_3 & m_4
\end{pmatrix}
\begin{pmatrix}
 j_1 & j_2 & j_3 & j_4 \\
 m_1 & m_2 & m_3 & m_4
\end{pmatrix}
$$

The orthogonality properties (7.7) of the generalized Clebsch-Gordan coefficients give us the following orthogonality properties of the generalized Wigner coefficients:

$$
\sum_{j_h, \ldots, j_k}
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 m_1 & \cdots & m_h & \cdots & m_k
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 -m_1 & \cdots & -m_h & \cdots & -m_k
\end{pmatrix}
$$

$$
= \delta(j_h, j_k) \delta_{j_h, j_k} \delta_{m_h, m_k} \left( -1 \right)^{j_h - m_h}
\times
\prod_{i=1}^{a-1} \delta(a_i, a_i') \delta(a_i, a_i')^{-1},
$$

(10.8a)

$$
\sum_{j_h, m_h}
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 m_1 & \cdots & m_h & \cdots & m_k
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 -m_1 & \cdots & -m_h & \cdots & -m_k
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 m_1 & \cdots & m_h & \cdots & m_k
\end{pmatrix}
$$

$$
= \prod_{i=k}^{j_h} \delta(m_i, m_i'),
$$

(10.8b)

If the first equation is also summed over $m_h$, putting $j_h' = j_h$ and $m_h' = m_h$ we obtain

$$
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 m_1 & \cdots & m_h & \cdots & m_k
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
 j_1 & \cdots & j_h & \cdots & j_k \\
 -m_1 & \cdots & -m_h & \cdots & -m_k
\end{pmatrix}
$$

$$
= 1,
$$

(10.9)

where the equalities $a_i' = a_i$ ($i = 1, \ldots, n - 3$) are assumed.
Multiplying (7.10) by the matrix element of the representation, integrating over the group and using (2.8), we obtain the formula

\[
\int \left( \frac{m_1}{D_{i_1}(\phi)} \frac{m_2}{D_{i_2}(\phi)} \cdots \frac{m_n}{D_{i_n}(\phi)} \right) \frac{df}{df} = 
\]

\[
= \sum_{i_1 \cdots i_n} (a_{ij}) \left( \begin{array}{c} j_1 \cdots j_n \\ m_1 \cdots m_n \end{array} \right)^A
\]

\[
\times \left( \begin{array}{c} j_1 \cdots j_n \\ m'_1 \cdots m'_n \end{array} \right)^A,
\]

which will be required later on (cf. section 13).

CHAPTER III

GRAPHICAL METHODS FOR OPERATIONS WITH SUMS OF PRODUCTS OF WIGNER COEFFICIENTS

In the solution of many problems there appear the sums of products of Clebsch-Gordan coefficients. It is convenient to replace them by the more symmetrical Wigner coefficients and work with sums of products of the latter. Like the Clebsch-Gordan coefficients, these coefficients represent the vector addition of two angular momenta. This suggests the possibility of a graphical representation of expressions containing Wigner coefficients. It might seem that the key geometrical figures in these graphical representations should be triangles representing the individual Wigner coefficients, as vector addition is geometrically represented by a triangle. However, this representation is inconvenient and has found no serious practical application.

It was found that a convenient representation of the Wigner coefficient is a point at which three lines corresponding to the three angular momenta converge. This is a satisfactory starting-point for devising graphical methods of operations with complicated expressions containing Wigner coefficients.

The graphical representation of trisads by points was used in the works of Ord-Smith /1954/ and Edmonds /1957/ for quantities which are independent of the projections (in which the projections have already been summed over). However, it was used only for this purpose and not in connection with sums of products of Wigner coefficients. In the works of Levinson /1957a, b, c/ a method was devised for replacing all algebraic calculations with Wigner coefficients by more convenient and general graphical transformations. This method is discussed in general terms in the present chapter.

We discuss the sums of products of Wigner coefficients in section 11 and their graphical representation in the next section. Section 13 is devoted to the expansion of these sums in generalized Wigner coefficients, and section 14 to the transformation of sums of products of Wigner coefficients. Finally, section 15 gives rules for the evaluation of expressions containing the sums of products of Wigner coefficients over angular momentum parameters.

11. Sums of products of Wigner coefficients (\(jm\)-coefficients)

We shall examine the products of Wigner coefficients which are summed over twice-repeated projections. We shall demand that the sum satisfy the following conditions:
a) The twice-repeated index of summation \( m \) should occur once with the positive sign and once with the negative sign, and in both cases be the projection of the same angular momentum \( j \).

b) Whenever the projection \( m \) of the angular momentum \( j \) is to be summed over, the summand should contain the multiplying factor \((-1)^{j-m}\).

The following is an example of a sum which satisfies these requirements:

\[
\sum_{n,n,n,n,n} (-1)^{j_{n}+j_{n}+j_{n}+j_{n}+j_{n}} \times \left( \begin{array}{c}
    j_{n} \\
    j_{n} \\
    j_{n} \\
    j_{n} \\
    j_{n}
\end{array} \right) \times \left( \begin{array}{c}
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n}
\end{array} \right) \times \left( \begin{array}{c}
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n}
\end{array} \right).
\]

The sums encountered in calculations may always be reduced to the form under consideration. Condition (a) may be satisfied by using the rule for the simultaneous change of sign of the magnetic quantum numbers in the Wigner coefficient, as given by (5.5). The phase factor required by (b) may be introduced under the summation sign from the equality \((-1)^{j_{m}} = (-1)^{j_{m}} \times (-1)^{j_{m}} \times (-1)^{j_{m}}\) and by using the fact that the sum of the three magnetic quantum numbers must vanish.

The sums of products of the Wigner coefficients may conveniently be called \( jm \)-coefficients, as they depend both on the angular momenta \( j \) and on those projections \( m \) which are not summed over. One such quantity is the expression (11.1), which depends on the angular momenta \( j_{1}, j_{2}, j_{3}, j_{4} \) (and is summed over the projections of these), and on the angular momenta \( m_{1}, m_{2} \) and their projections \( m_{3}, m_{4} \) which are not summed over. It is convenient to write this \( jm \)-coefficient in the following manner:

\[
\left( \begin{array}{c}
    j_{1} \\
    j_{2} \\
    j_{3} \\
    j_{4}
\end{array} \right) \times \left( \begin{array}{c}
    m_{1} \\
    m_{2} \\
    m_{3} \\
    m_{4}
\end{array} \right).
\]

The angular momenta whose projections are summed over may conveniently be grouped into sets. Various criteria may be adopted for this grouping. Thus in (11.2) they are grouped in such a way that the first \( I \) forms a triad with each of the following two pairs \((l_{1}l_{2}l_{3})\) and \((l_{1}l_{2}l_{3})\), respectively, and in the three pairs \((l_{1}l_{2}l_{3}, l_{1}l_{2}l_{3})\), \((l_{1}l_{2}l_{3}, l_{1}l_{2}l_{3})\), \((l_{1}l_{2}l_{3}, l_{1}l_{2}l_{3})\), the first and the second members are taken together, that is \((l_{1}l_{2}l_{3})\) and \((l_{1}l_{2}l_{3})\), also form triads. However, one cannot write different sums in the same form, as they differ essentially from each other. Therefore for each case one must choose a distribution of parameters which will allow a compact notation and will represent the triad structures as clearly as possible. Thus if we take the sum

\[
\sum_{n,n,n,n,n} (-1)^{j_{n}} \left( \begin{array}{c}
    j_{n} \\
    j_{n} \\
    j_{n} \\
    j_{n} \\
    j_{n}
\end{array} \right) \times \left( \begin{array}{c}
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n}
\end{array} \right) \times \left( \begin{array}{c}
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n} \\
    m_{n}
\end{array} \right),
\]

(11.3)

This is easily seen by introducing (4.9b) into (3.3). Using (4.9b) and (6.5) one can easily verify that

\[
\sum_{m} (-1)^{j_{m}} \psi (j_{m}) \psi (jm) = \left( \begin{array}{c}
    j_{m} \\
    j_{m}
\end{array} \right) \times \left( \begin{array}{c}
    m_{m} \\
    m_{m}
\end{array} \right) \times \left( \begin{array}{c}
    m_{m} \\
    m_{m}
\end{array} \right).
\]

a clear way of writing it would be

\[
F \left( \begin{array}{c}
    j_{1} \\
    j_{2} \\
    j_{3} \\
    j_{4}
\end{array} \right) \times \left( \begin{array}{c}
    m_{1} \\
    m_{2} \\
    m_{3} \\
    m_{4}
\end{array} \right) \times \left( \begin{array}{c}
    m_{1} \\
    m_{2} \\
    m_{3} \\
    m_{4}
\end{array} \right).
\]

(11.4)

since from this the distribution of angular momenta in triads can be easily seen. However, for compactness it may be written as

\[
F \left( j_{1}j_{2}j_{3}j_{4} \right) \times \left( \begin{array}{c}
    m_{1} \\
    m_{2} \\
    m_{3} \\
    m_{4}
\end{array} \right) \times \left( \begin{array}{c}
    m_{1} \\
    m_{2} \\
    m_{3} \\
    m_{4}
\end{array} \right).
\]

(11.5)

It would be obvious that the generalized Wigner coefficients are also \( jm \)-coefficients. They represent those \( jm \)-coefficients which, for a given number of free angular momenta, have a minimum number of contracted angular momenta. The notation for the generalized Wigner coefficients is given in section 10.

It should be noted that in the particular case where all magnetic quantum numbers are summed over, the resulting expression is independent of these. These quantities are called \( j \)-coefficients and are studied in the next chapter.

The form of summation which we have chosen for products of Wigner coefficients is closely related to the invariance of the expression

\[
\sum_{m} (-1)^{j_{m}} \psi (j_{m}) \psi (jm) = \left( \begin{array}{c}
    j_{m} \\
    j_{m}
\end{array} \right) \times \left( \begin{array}{c}
    m_{m} \\
    m_{m}
\end{array} \right) \times \left( \begin{array}{c}
    m_{m} \\
    m_{m}
\end{array} \right).
\]

(11.6)

This is easily seen by introducing (4.9b) into (3.3). Using (4.9b) and (6.5) one can easily verify that

\[
\sum_{m} \psi (j_{1}m_{1}) \psi (j_{2}m_{2}) \psi (j_{3}m_{3}) \psi (j_{4}m_{4}) \psi (j_{1}j_{2}j_{3}j_{4}) = \left( \begin{array}{c}
    j_{1} \pm j_{2} \pm j_{3} \pm j_{4}
\end{array} \right) \times \left( \begin{array}{c}
    m_{1} \pm m_{2} \pm m_{3} \pm m_{4}
\end{array} \right).
\]

(11.7)

i.e., is also invariant. Equation (11.6) shows that when the Wigner coefficients are contracted the role of the Kronecker delta is taken over by the quantity

\[
\delta (j_{1}m_{1}, m_{1}) = (-1)^{j_{m}} \delta (m_{1}, m_{1}) = (-1)^{j_{m}} \delta (m_{1}, m_{1})
\]

(11.8)

This is also clear from the following

\[
\sum_{m} (-1)^{j_{m}} \psi (j_{m}k) \psi (j_{1}m_{1}q_{1}) \psi (j_{1}m_{1}q_{1}) = \sum_{m} (-1)^{j_{m}+j_{1}+q_{1}} \psi (j_{1}m_{1}q_{1}) \times \left( \begin{array}{c}
    j_{1} \pm j_{1} \pm q_{1}
\end{array} \right) \times \left( \begin{array}{c}
    m_{1} \pm m_{1} \pm q_{1}
\end{array} \right).
\]

(11.9)
\[
\sum_i T^{\text{im}} U_i^{\text{mT}} = \sum_i T^{\text{im}} U_i^{\text{mT}} b_i^j.
\] (11.10)

From (11.7) it may be seen that the Wigner coefficient is the analogue of the fully antisymmetric tensor \( \epsilon_{m_n} \), with which one forms the scalar triple product of three vectors. For the sake of generality it is convenient to consider sums containing, in addition to Wigner coefficients, the quantities
\[
\begin{pmatrix} i_1 & i_2 \\ m_1 & m_2 \end{pmatrix} = \begin{pmatrix} i_1 & i_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} = (j_1) \cdot \delta (j_1, j_2) (-1)^{i_1+i_2} \delta (m_1, -m_2)
\] (11.11)

and
\[
\begin{pmatrix} i_1 & i_2 \\ m_1 & 0 \end{pmatrix} = \begin{pmatrix} i_1 & 0 & 0 \\ m_1 & 0 & 0 \end{pmatrix} = \delta (j_1, 0) \delta (m_1, 0),
\] (11.12)
as well as the delta \( \delta (-m, m') \), although these sums can be simplified in an elementary manner owing to the presence of the Kronecker deltas in (11.8), (11.11) and (11.12).

It will be useful to remember that
\[
\begin{pmatrix} i_1 & i_2 \\ m_1 & m_2 \end{pmatrix} = (-1)^{i_1+i_2} \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}
\] (11.13)

analogous to (5.4). We note that the triangular delta may also be regarded as a \( f \)-coefficient, as from (5.6a) it follows that
\[
\begin{pmatrix} i_1 & i_2 & i_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sum_{m_1m_2m_3} (-1)^{i_1+i_2+i_3-m_1-m_2-m_3} \times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
\] (11.14)

12. Graphical representation of \( jm \)-coefficients

When dealing with the sums of products of Wigner coefficients it is convenient to make use of diagrammatic representations. The elements of these diagrams should be diagrams of Wigner coefficients. The Wigner coefficient \( \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \) is represented by an oriented node which links three directed lines. These lines (Figure 12.1) correspond to \( j_1 \), \( j_2 \), and \( j_3 \) and their free ends to the projections \( m_1 \), \( m_2 \) and \( m_3 \). By the orientation of the node is meant the cyclic order of the lines \( j_1 \), \( j_2 \), \( j_3 \), which is shown in the diagram by circular arrows. The direction of the line \( j_1 \) indicates the sign of the magnetic quantum number \( m_1 \) in the Wigner coefficient.

A line directed away from the node corresponds to the positive sign of the magnetic quantum number, and a line directed towards the node to the negative sign. Thus the Wigner coefficient \( \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & -m_2 & -m_3 \end{pmatrix} \) may be represented by the diagram of Figure 12.2. Apart from rare exceptions, the magnetic quantum numbers will not be indicated in the diagrams. We shall adopt the convention that the projections of the angular momenta \( j \), \( l \), \( k \) and \( a \) will be respectively \( m \), \( n \), \( q \) and \( \mu \), with corresponding indices.

The quantities \( \begin{pmatrix} i_1 & i_2 & i_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \) and \( \begin{pmatrix} i_1 & i_2 \\ m_1 & m_2 \end{pmatrix} \) may be represented analogously. Their diagrams are shown in Figures 12.3 and 12.4 respectively. In Figure 12.3 the circular arrow indicates the order of succession of the angular momenta \( j_1 \) and \( j_2 \). The signs of \( m_1 \) and \( m_2 \) are indicated in the same way as above.

Figure 12.5 shows the diagrammatic representation of \( \begin{pmatrix} j & -m \end{pmatrix} \). This diagram is the directed line of the angular momentum \( j \), the end of which corresponds to the projections \( m \) and \( m' \). The signs of \( m \) and \( m' \) in \( \delta \) \( j \) are again indicated as above. The direction of the line \( j \) may be changed, which corresponds to the symmetry of \( j \) in \( m \) and \( m' \).

Figure 12.4 Figure 12.5

In practice one rarely encounters the representations of the quantities \( \begin{pmatrix} i_1 & i_2 \\ m_1 & m_2 \end{pmatrix} \) and \( \begin{pmatrix} i_1 & i_2 \\ m_1 & m_2 \end{pmatrix} \). It is therefore more convenient to denote the orientation of the node representing the Wigner coefficient by a sign. A clockwise orientation is denoted by a \( + \) sign and a counter-clockwise orientation by a \( - \) sign (Figure 12.6).

Rotation of the diagram does not change the cyclic order of the lines. Owing to the symmetry (5.4) a Wigner coefficient remains unchanged by a cyclic permutation of the columns. Therefore a rotated diagram represents the same Wigner
coefficient as the initial diagram. The angles between the lines as well as their lengths have no significance. Consequently, geometrical deformations of a diagram which preserve the orientation of the node do not change the Wigner coefficient represented by the diagram. Deformations which change the cyclic order of the lines around the node imply a change in the orientation of the node. In accordance with (5.4), the Wigner coefficient represented by such a deformed diagram will differ from that represented by the original diagram by the factor $(-1)^{s+sh}$. In many cases such a change in the orientation of the node may be compensated by a change in the sign of the node, as changing the sign amounts to the same thing as changing the orientation.

Summation over a magnetic quantum number $m$ is graphically represented by joining the free ends of the corresponding lines*. As an example, let us consider

Figure 12.7

the sum in the left-hand side of (11.9). Figure 12.7 shows the diagrams of the Wigner coefficients which appear in the sum. Joining the free ends which correspond to the magnetic quantum number $m$ we obtain the diagram shown in Figure 12.8. This diagram represents the sum under consideration. Let us now consider the sum in the right-hand side of (11.9). In this case we have to represent the two Wigner coefficients and the delta, and join the free ends $m, m'$ and $m', m'$ (Figure 12.9). The diagram thus obtained is identical with Figure 12.8.

The diagrams $A$ and $F$ similarly obtained for (11.1) and (11.3) are shown in Figures 12.10 and 12.11 respectively.

* The remainder of the book leans heavily on the diagrammatic representations. For a description of the diagrams and of operations with them we introduce the following systematic nomenclature:

1. A line with a free end will be called a "free" line.
2. A diagram which contains free lines will be called an "open diagram"; a diagram which contains no free lines will be called a "closed" diagram.
3. If two open diagrams are such that each contains a free line referring to the same angular momentum $j$, these two free lines will be called "corresponding" free lines.
4. Joining the free ends of two lines referring to the same angular momentum will be called "contracting" the lines. Two lines may be contracted only when they are similarly directed. This can always be arranged, as it involves only a change in the phase factor.
5. Contracting all pairs of corresponding free lines in two open diagrams will be called "contracting" the two diagrams.

Translator.
Wigner coefficient for the scheme of addition \((1 + 2) + (3 + 4)\) and Figure 12.14 the same for the scheme \((1 + 2) + (3 + 4)\). Comparing these diagrams we see that these generalised Wigner coefficients are essentially similar. Therefore the generalised Clebsch-Gordan coefficients for the schemes of addition (10.6) and (10.7) are expressed in terms of the same generalised Wigner coefficient.

For any scheme of addition whatsoever, the diagrams of the generalised Wigner coefficients do not contain closed figures. This follows from the fact that the corresponding sums contain the minimum number of contracted angular momenta. It is convenient to give a branching form to the generalised Wigner coefficients, as is done in Figures 12.13 and 12.14. Obviously, any diagram which does not contain closed figures and which is represented in branching form may be related to some generalised Wigner coefficient.

In many cases the structure of a part of the diagram and not of the whole is of importance. In these cases the remaining parts of the diagram may be regarded as arbitrary and replaced by one or several blocks. Thus, if we wish to separate the line \(j\) which joins two nodes in a diagram, we may do so by means of the block diagram \(W\) in Figure 12.15. The block \(a\) replaces all other lines in the diagram together with the scheme of joining these lines to the lines \(j_1, j_2, j_3\) and \(j_4\). The lines in the block \(a\) may include free lines as well as those which join two nodes. If the block consists only of lines of the second kind it is referred to as 'closed'. In general the block will be 'open'. The block diagram in Figure 12.16 indicates that the diagram \(F\) contains in all four free lines \(j_1, j_2, j_3\), and \(j_4\). The remaining lines each join two nodes and the set of these lines is replaced by the closed block \(\bar{a}\) (the bar above indicates a closed block). The blocks are usually separated when they are not important for further transformations with diagrams or \(jm\)-coefficients. We shall therefore denote them by subscripts in the \(jm\)-coefficients. Thus the \(jm\)-coefficients corresponding to the diagrams \(W\) and \(F\) may be written as follows:

\[
W_a(j_1, j_2, j_3, j_4)
\]  \hspace{1cm} (12.2)

\[
F_{\bar{a}}(j_1, j_2, j_3, j_4)
\]  \hspace{1cm} (12.3)

A block diagram may also contain several blocks.

When constructing the diagram for a \(jm\)-coefficient one chooses the simplest and most explicit form, since every change (deformation) of a diagram which does not change the orientations of its nodes and the directions of its lines will not affect the \(jm\)-coefficient represented by the diagram. We note that diagrams containing nodes with two lines can easily be simplified. Comparing the diagrams (12.2) and (12.3) with the corresponding quantities (11.11) and (11.8) we see that if a line \(j_1\) issues from a node and a line \(j_1\) converges to it, and if the orientation of the node is from the former to the latter, then the node may be dropped. This gives the factor \(\delta(j_1, j_2, j_3)\). If a parameter \(j\) is zero, the corresponding line is simply erased.

The triangular delta, as may be seen from (11.14), is represented by the diagram of Figure 12.17.

13. Expansion of \(jm\)-coefficients in generalised Wigner coefficients

If a \(jm\)-coefficient is not a generalised Wigner coefficient (i.e., if the number of contracted angular momenta is not minimal), it may then be expanded in generalised Wigner coefficients. Let us consider an arbitrary \(jm\)-coefficient, the diagram of which contains the free lines \(j_1, j_2, \ldots, j_n\) to which correspond the projections \(m_1, m_2, \ldots, m_n\). All the remaining lines \(i_1, i_2, \ldots, i_n\) join nodes. We shall replace the set of these by the closed block \(\bar{a}\) and denote the \(jm\)-coefficient under consideration by

\[
F_{\bar{a}}(j_1, j_2, j_3, \ldots, j_n)
\]

Let us now express \(F\) as the sum of products of Wigner coefficients and substitute an equivalent expression for each of the latter in accordance with (5.10). The matrix elements of the representations corresponding to the angular momenta \(i_k\)
can be left out. To prove this, consider an angular momentum which appears twice, once with the projection \( n \) and once with the projection \(-n\). The part of the sum which contains this parameter is therefore

\[
\sum_n (-1)^{y-n} \binom{i}{\ldots \cdot n} \binom{i}{\ldots \cdot -n}, \tag{13.1}
\]

where the dots indicate other parameters which are of no interest for the present. After introducing (5,10) we will have, instead of (13.1), the expression

\[
\sum_{n,n'} (-1)^{y-n'} \binom{i}{\ldots \cdot n'} \binom{i}{\ldots \cdot n} \times
\]

\[
x \left( |n'| D_1(n') | n \right) \left( |n'| D_2(n) | -n \right). \tag{13.2}
\]

The summation over \( n \) is carried out with the aid of the unitarity relation (2,5a) of the matrix \( D_1 \); the resulting expression is

\[
\sum_{n,n'} \binom{i}{\ldots \cdot n'} \binom{i}{\ldots \cdot n} (-1)^{y-n'} \delta(-n', n'). \tag{13.3}
\]

This sum is identical with (13.1), thus proving the assertion. Therefore, after introducing (5,10) one may retain only the matrix elements of those representations which correspond to the parameters \( j_i \), i.e.,

\[
F_R \left( j_i \cdots j_n \right) = \sum_{m_1, \ldots, m_n} \delta \left( j_i \cdots j_n \right) \binom{i}{m_1 \cdots m_n} \binom{i}{m_1 \cdots m_n}, \tag{13.4}
\]

Let us first consider the general case \( n > 3 \). In this case we integrate the above equation over the group, using (10,10). We then obtain the required expansion of the jm-coefficient in generalized Wigner coefficients

\[
F_R \left( j_i \cdots j_n \right) = \sum_{a, \ldots, a_{n-3}} (a_1, \ldots, a_{n-3}) \sum_{m_1, \ldots, m_n} \delta \left( j_i \cdots j_n \right) \binom{i}{m_1 \cdots m_n}, \tag{13.5}
\]

where the expansion coefficient is

\[
R_R (a_1, \ldots, a_{n-3}; j_i \cdots j_n) = \sum_{m_1, \ldots, m_n} F_R \left( j_i \cdots j_n \right) \binom{i}{m_1 \cdots m_n}. \tag{13.6}
\]

In order to reduce the summation over the magnetic quantum numbers in this expression to the adopted standard form, one must make use of (10,5) and of the vanishing of the sum of all the projections in the generalized Wigner coefficient. We then have

\[
R_R (a_1, \ldots, a_{n-3}; j_i \cdots j_n) = \sum_{m_1, \ldots, m_n} (-1)^{y=-m_1+\ldots+n-n} \times
\]

\[
\times F_R \left( j_i \cdots j_n \right) \binom{i}{m_1 \cdots m_n} \binom{i}{m_1 \cdots m_n} \times \tag{13.7}
\]

From here it is seen that \( R \) is a jm-coefficient, the diagram of which may be obtained by contracting the diagram \( F \) and the diagram of the generalized Wigner coefficient, retaining for the contracted lines their directions in \( F \). We note that the form of the contraction \( R \) therefore depends on the generalized Wigner coefficient in which the expansion is carried out. It follows from the expansion (13.5) that a jm-coefficient will not vanish only when the free angular momenta satisfy the polygon conditions and when the sum of their projections is zero. Otherwise, the generalized Wigner coefficient appearing in the right-hand side of (13.5) will vanish. It should be noted that for \( n = 3 \) the summation in (13.5) is trivial. In this case we have

\[
F_R \left( j_1 \ j_2 \ j_3 \right) = R_R \left( j_1 \ j_2 \ j_3 \right) \left( j_1 \ j_2 \ j_3 \right). \tag{13.8}
\]

where

\[
R_R \left( j_1 \ j_2 \ j_3 \right) = \sum_{m_1, m_2, m_3} (-1)^{y=m_1-m_2+m_3} \times
\]

\[
\times F_R \left( j_1 \ j_2 \ j_3 \right) \binom{i}{m_1 \cdots m_3}, \tag{13.9}
\]

The diagram of the coefficient \( R \) is obtained simply by joining the free lines in diagram \( F \) at a single node. The orientation of this node should be identical with that of the node which represents the Wigner coefficient chosen in (13.8).

For \( n = 2 \) the generalized Wigner coefficient, both in (13.5) and in (13.6), becomes the particular case (5.7) of the usual Wigner coefficient, so that the two parameters \( j_i \) and \( j_n \) must form a "polygon". As a result we obtain

\[
F_R \left( j_1 \ j_n \right) = R_R \left( j_1 \ j_n \right) \left( j_1 \ j_n \right) = \delta \left( j_1, j_n \right), \tag{13.10}
\]

The diagram of the \( j \)-coefficient \( R_R (j_1 j_n) \) is obtained simply by joining the two free lines in diagram \( F \) at one node. The diagram of the \( j \)-coefficient \( R_R (j_1 j_n) \) is obtained by joining the lines \( j_1 \) and \( j_n \) in diagram \( F \) to form a single line with the direction of \( j_1 \).
For $n = 1$ we have

$$F_4 \left( \frac{i_1}{m_1} \right) = R_4 \left( \frac{j_1}{m_1} \right) =
\begin{array}{c}
3 \left( j_1, 0 \right) b \left( m_1, 0 \right) F_3 \left( \frac{0}{0} \right).
\end{array}$$

The diagram of $R_4 \left( j_1 \right)$ is obtained by adding a node to the free end of the line $j_1$ in diagram $F_3$.

14. Transformation of jm-coefficients

The formula for the expansion of jm-coefficients in generalized Wigner coefficients which was established in the preceding section allows one to obtain a very general method of expressing jm-coefficients in terms of coefficients with a smaller number of parameters; this method makes possible a clear graphical interpretation.

We shall say that the diagram $G$ is “separable” on $n$ lines $l_1, l_2, \ldots, l_n$ when, if these lines are “cut”, the diagram $G$ breaks up into two diagrams $A$ and $B$, one of which, say $B$, contains all the free lines in diagram $G$. Further, in nontrivial cases, diagrams $A$ and $B$ will contain more than one node. Thus diagram $G$ in Figure 14.1 is separable on the four lines $l_1, l_2, l_3, l_4$, the four lines $l_1, l_2, l_3, l_4$, and the two lines $s_2, s_4$. However, it will not be separable on the four lines $l_1, l_2, p, s$, as both parts then formed will contain free lines from diagram $G$ (the upper part the line $j_1$ and the lower part the line $j_3$). If the diagram $G$ is a diagram of a j-coefficient, i.e., does not contain free lines, the condition of separability reduces to the decomposition of the diagram after separation into two parts.

In the general case the jm-coefficient $G$, which is separable on the lines $l_1, l_2, \ldots, l_n$, may be written as the following algebraic expansion:

$$G_{j_n} (l_1, \ldots, l_n) = \sum_{n, m, n} (-1)^{y-n+n+1+n-s-n} \times$$

$$\times A_{j_n} \left( \frac{l_1, \ldots, l_n}{n_1, \ldots, n_n} \right) B_n \left( l_1, \ldots, l_n \right).$$

(14.1)

We have initially oriented the line $l_1$ in the direction $A$ to $B$. The block $A$ is closed, as the only free lines in diagram $A$ are the lines $l_1, \ldots, l_n$.

Let us apply formula (13.5) to the jm-coefficient $A$ in (14.1). We obtain the following equation

$$G_{j_n} (l_1, \ldots, l_n) = \sum_{a_1, \ldots, a_n} (a_1, \ldots, a_n) R_{a_1} (a_1, \ldots, a_n; l_1, \ldots, l_n) \times$$

$$\times C_{j_n} (a_1, \ldots, a_n; l_1, \ldots, l_n),$$

(14.2)

where the j-coefficient

$$R_{a_n} (a_1, \ldots, a_n; l_1, \ldots, l_n) = \sum_{n_1, \ldots, n_n} (-1)^{y-n+n+1+n-s-n} \times$$

$$\times A_{a_n} \left( \frac{l_1, \ldots, l_n}{n_1, \ldots, n_n} \right) \left( l_1, \ldots, l_n \right) A_n \left( n_1, \ldots, n_n, \ldots, n_{n-1} \right).$$

(14.3)

and the jm-coefficient

$$C_{j_n} (a_1, \ldots, a_n; l_1, \ldots, l_n) = \sum_{n_1, \ldots, n_n} (-1)^{y-n+n+1+n-s-n} \times$$

$$\times B_{a_n} \left( \frac{l_1, \ldots, l_n}{n_1, \ldots, n_n} \right) \left( l_1, \ldots, l_n \right) B_n \left( n_1, \ldots, n_n, \ldots, n_{n-1} \right).$$

(14.4)

Thus, formula (14.2) expresses the jm-coefficient $G$, which is separable on $n$ lines, in the form of an $(n-3)$-fold sum of the product of the j-coefficient $R$ and the jm-coefficient $C$, each of which contains fewer parameters than the initial jm-coefficient $G$. From (14.3) and (14.4) it is seen that diagrams $R$ and $C$ are obtained by contracting the respective separated diagrams with the diagram of the chosen generalized Wigner coefficient. By changing the diagram of this coefficient one may choose the most suitable form for the coefficients $R$ and $C$.

The jm-coefficient $G$, which is separable on $n$ lines $l_1, l_2, \ldots, l_n$, tends to zero if the parameters $l_1, l_2, \ldots, l_n$ do not satisfy the polygon condition. Clearly, when this condition is not satisfied the generalized Wigner coefficients in (14.3) and (14.4) vanish, and therefore $R$ and $C$ vanish, together with $G$.

Formula (14.2) is particularly important for $n = 3$, as in this case there is no summation over $a$:

$$G_{j} (l_1, l_2, l_3) = R_{j} (l_1, l_2, l_3) C_{j} (l_1, l_2, l_3),$$

(14.5)

where the j-coefficient

$$R_{j} (l_1, l_2, l_3) = \sum_{n_1, n_2} (-1)^{y-n+n+1+n-s-n} \times$$

$$\times A_{j} \left( \frac{l_1, l_2, l_3}{n_1, n_2} \right) \left( l_1, l_2, l_3 \right) A_{n_1, n_2} \left( n_1, n_2, n_3 \right).$$

(14.6)

and the jm-coefficient

$$C_{j} (l_1, l_2, l_3) = \sum_{n_1, n_2} (-1)^{y-n+n+1+n-s-n} \times$$

$$\times B_{j} \left( \frac{l_1, l_2, l_3}{n_1, n_2, n_3} \right) \left( l_1, l_2, l_3 \right) B_{n_1, n_2, n_3} \left( n_1, n_2, n_3 \right).$$

(14.7)
Formula (14.5) shows that a \( jm \)-coefficient which is separable on three lines decomposes into the product of a \( j \)-coefficient and a \( jm \)-coefficient, each of which has a smaller number of parameters. To obtain the diagrams of these coefficients one must, in each part, join the three free lines obtained upon separation at a single node, with the same orientation in the two parts.

A unique case arises for \( n = 2 \). The diagram of the coefficient \( H \), which is separable on two lines \( (l_1 \) and \( l_2) \), is represented in Figure 14.2. In this case the \( jm \)-coefficient \( H \) reduces to the product of the \( j \)-coefficient \( R \) and the \( jm \)-coefficient \( C \).

Their diagrams are obtained by cutting the lines \( l_1 \) and \( l_2 \) in diagram \( H \) and joining the free lines of each part at one node. As both diagrams contain a node with two lines, they may be simplified, and we obtain

\[
H_{ij} (l_1, l_2) = (-1)^{j_1} b (l_1, l_2) \times A_2 (l_1) B_2 (l_2).
\]

(14.8)

A particular case of this formula is the rule of simplification of the diagrams of \( jm \)-coefficients in which two nodes are joined by two lines (Figure 14.3). We first change the direction of the line \( l_1 \) and make use of (14.8). Diagram \( B \) then reduces to the diagram of a triangular delta (Figure 12.17). Thus we have

\[
F_{ij} = b (l_1, l_2) a (l_1, l_2) A_2 (l_1, l_2) B_2 (l_1, l_2).
\]

(14.9)

If the diagram is separable on the single line \( l_1 \) (Figure 14.2, in which the line \( l_2 \) is missing), we obtain analogously

\[
H_{ij} (l_1) = b (l_1, 0) A_2 (l_1, 0) B_2 (l_1, 0).
\]

(14.10)

Let us use the formula obtained to simplify the diagram in which a line issues and ends on the same node (Figure 14.4). Separating the diagram \( F \) on \( l \), we have

\[
F_{ij} = b (l, i) a (l, i) A_2 (l, i) B_2 (l, i)\Phi_a.
\]

(14.11)

where \( \Phi_a \) denotes the \( jm \)-coefficient represented in Figure 14.4.

As we have seen, all cases of diagrams separable on \( n \leq 3 \) lines enable the \( jm \)-coefficients to be simplified, by expressing them in terms of products of simpler coefficients. For \( n = 2 \) and 1 one of these coefficients becomes a delta, as we have seen in the particular cases.

To conclude the present section, we give the rule for the simplification of diagrams of \( jm \)-coefficients for the case where the value of one of the contracted angular momenta is zero. Let us consider the \( jm \)-coefficient \( W_{ij} (j, i; j, i; j) \) (diagram in Figure 14.5). If \( j = 0 \) the line \( j \) is dropped, as indicated in section 12. Two nodes, each with two lines, are then obtained. Dropping these nodes we find

\[
W_{ij} (j, i; j, i; 0) = (\alpha) (\beta) b (j, i) a (j, i) U_a (j, i).
\]

(14.12)

One can see from diagram 14.5 how diagram \( U \) is obtained from the initial diagram \( W \).
15. Summation of \( jm \)-coefficients

In calculations with \( jm \)-coefficients one is often required to sum the products of these coefficients over the parameters \( j \). In this section we shall indicate how this summation may be carried out graphically with the help of diagrams. We shall state the problem as follows: given the diagrams of the coefficients which are summed over, we wish to find the diagram of the coefficient which is the given sum.

The diagram of the sum may be found from two rules which admit of simple graphical interpretation. The first is the "summation rule" of the \( jm \)-coefficient over the parameters \( j \). Consider the diagram \( C \) in Figure 15.1. The corresponding \( jm \)-coefficient may be written as follows:

\[
C_a(j_1 j_1'; j_2 j_2'; a) = \sum_{m_1 m_1'} (-1)^{j_1'' - m_1 - j_1'' - m_1'} A_{a}(j_1 j_2; j_1' j_2') A_{a}(j_1 j_2; j_1' j_2').
\]

(15.1)

For \( j_1 = j_1' \) and \( j_2 = j_2' \) the \( jm \)-coefficient \( C \) may be summed over \( a \) with the aid of (5.6b):

\[
\sum_a C_a(j_1 j_1'; j_1' j_1'; a) = B_a(j_1 j_1').
\]

(15.2)

where

\[
B_a(j_1 j_1') = \sum_{m_1 m_1'} (-1)^{j_1'' - m_1} A_{a}(j_1 j_1; j_1' j_1').
\]

(15.3)

The diagram of the \( jm \)-coefficient \( B \) is given in Figure 15.1. By comparing diagrams \( C \) and \( B \) one can formulate a convenient graphical rule for the summation of \( jm \)-coefficients over angular momentum quantum numbers. Let the nodes at the ends of the line representing the summation parameter \( j \) join corresponding lines and have the same orientation, and let one node join converging lines and the other diverging lines. To find the diagram of the sum one must cut the lines adjacent to the line \( j \), drop the part of the diagram containing \( j \) and contract corresponding lines in the remaining part.

The second rule is the "rule of multiplication" of a \( jm \)-coefficient and a \( j \)-coefficient. Let us consider the product of the \( jm \)-coefficient \( C_b(l_1 l_1'; b) \) and the \( j \)-coefficient \( R_a(l_1 l_1'; b) \). The diagram of this \( jm \)-coefficient may be obtained by erasing identical nodes from diagrams \( C \) and \( R \), followed by contraction of the two diagrams.

With these two rules summations may be carried out graphically in all cases. However, as to avoid repeated application of the multiplication rule, it is more practical to derive a general rule for the summation of the products of coefficients over single angular momentum parameters.

Consider the sum of the products of the \( jm \)-coefficients \( C_a \) and the \( j \)-coefficients \( W_{\bar{a}} \) (\( i = 1, 2, \ldots, \bar{a} - 1 \)):

\[
A = \sum C_a(j_1 j_1'; j_2 j_2'; a) W_{\bar{a}}(j_3 j_3'; j_4 j_4'; a) \cdots
\]

(15.4)

The diagrams of \( C_a \) and \( W_{\bar{a}} \) are given in Figure 15.3. It is obvious that this representation of the diagrams is equivalent to the representation in Figure 15.1. The diagrams \( C_a \) and \( W_{\bar{a}} \) have identical and identically oriented nodes joining the lines \( j_1, j_2, a \). In accordance with the "multiplication rule", the product \( C_a W_{\bar{a}} \) may be expressed in the form of the \( jm \)-coefficient \( C_{a, \bar{a}} \). The diagram of this \( jm \)-coefficient is given in Figure 15.4. One sees from this figure that the \( jm \)-coefficient \( C_{a, \bar{a}} \) may be represented in the form \( C_{a, \bar{a}}(j_1 j_1'; j_2 j_2'; a) \), where the block \( a \) is obtained by joining the lines \( j_1' \) and \( j_2' \) in the blocks \( a \) and \( \bar{a} \). Continuing this
process, we can write the product under the summation sign in (15.4) as \( C_{\alpha_1 \cdots \alpha_n} \). The summation of this \( jm \)-coefficient over \( \alpha \) may be carried out by the "summation rule" (15.2). The diagram of the sum \( \mathbf{A} \) will be of the form shown.

In Figure 15.5, the rule for obtaining this diagram may be formulated as follows. Cut the lines adjacent to the lines \( \alpha \) in the individual diagrams, drop the parts containing the lines \( \alpha \) and contract corresponding lines in the remaining parts. Further, the lines and nodes adjacent to the lines \( \alpha \) must be oriented correspondingly.

These rules of summation make it possible to sum any expression over angular momentum parameters. Concrete cases of such summations will be discussed later on (see Chapter VI).

Chapter IV

\( j \)-coefficients and their properties

In the mathematical apparatus of the theory of angular momentum an important part is played by products of Wigner coefficients which are summed over all magnetic quantum numbers. As noted in the preceding chapter, these quantities are called \( j \)-coefficients. In the sense of (11.6), they are invariant under rotation of the coordinate system. As we saw in section 13, they are used to express the coefficients of expansion of an arbitrary sum of products of Wigner coefficients in terms of generalized Wigner coefficients.

It is easily seen that the diagram of a \( j \)-coefficient, which does not have free lines, contains \( 2n \) nodes and \( 3n \) lines \((n = 1, 2, 3, \ldots)\). The \( j \)-coefficient thus contains \( 3n \) parameters. The number of different \( j \)-coefficients with a given number of parameters \( 3n \) is determined by the number of different diagrams with \( 2n \) nodes. Further, only those diagrams which are separable on no less than four lines are significant, as from section 14 the \( j \)-coefficients represented by the other diagrams reduce to products of simpler \( j \)-coefficients. For given \( n \) all significant diagrams may be constructed either directly /Gutman and Budirte 1959/ or by recurrence from diagrams for \( n = 1 \)/Levinson and Chipur 1958/.

For \( n = 1 \) we have the trivial \( j \)-coefficient, the triangular delta. For \( n = 2 \) there is one \( j \)-coefficient; it is the simplest nontrivial \( j \)-coefficient, and is described in section 16. The number of essentially different \( j \)-coefficients for \( n = 3 \), \( 4, 5 \), and \( 6 \) is 1, 2, 5 and 18 respectively.

By separating the diagram of an arbitrary \( j \)-coefficient, one can represent it in the form of a (generally) multiple sum of \( 6j \)-coefficients. From the diagrams it is easily seen that single sums of this type (section 17) can only be of two kinds. Consequently, \( j \)-coefficients can be expressed in the form of single sums only for \( n = 3 \) and \( 4 \) (sections 18 and 19). When \( j \)-coefficients with more than twelve parameters are considered, multiple sums make their appearance (section 20 and appendices 3 and 4).

16. The \( 6j \)-coefficient and its properties

The \( 6j \)-coefficient, frequently called the Racah coefficient, is a product of four Wigner coefficients summed over all the magnetic quantum numbers (the form given below was obtained by Wigner /1937/; the \( W \)-coefficient examined by Racah /1942/ differs from the following by a phase factor (see appendix 1)).
\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= \sum_{m}\left( -1 \right)^{j_1 - m_1 + j_2 - m_2 + j_3 - m_3 + h_1 - n_1 + h_2 - n_2 + h_3 - n_3} \\
&\times \left( \begin{array}{ccc}
j_1 & l & j_3 \\
m_1 & l & m_3 \\
- m_1 & n_1 & - m_2 \\
- n_1 & - n_2 & m_3 \\
\end{array} \right) \left( \begin{array}{ccc}
j_1 & l & j_2 \\
- m_1 & n_1 & - m_2 \\
- n_1 & - n_2 & m_3 \\
\end{array} \right) \left( \begin{array}{ccc}
l & h & l \\
- m_1 & n_1 & - m_2 \\
- n_1 & - n_2 & m_3 \\
\end{array} \right) \\
&\times \left( \begin{array}{ccc}
l & h & l \\
- m_1 & n_1 & - m_2 \\
- n_1 & - n_2 & m_3 \\
\end{array} \right) \\
\end{align*}
\]

(16.1)

The graphical representation of this coefficient is easily obtained from the rules derived in section 12. It is given in Figure 16.1.

![Figure 16.1](image)

Racah (1942) reduced the expression (16.1) for the $6j$-coefficient to a single sum and obtained the following formula:

\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= (-1)^{j_1 + j_2 + j_3} \Delta(j_1 j_2 j_3) \Delta(j_1 j_3 j_4) \Delta(j_2 j_4 j_1) \Delta(j_3 j_1 j_2) \\
&\times \sum_{\sigma} (-1)^{j_1 + j_2 + j_3 + l_4 + \sigma} \left( \begin{array}{ccc}
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\end{array} \right) \\
&\times \frac{1}{(j_1 + j_2 + j_3 + l_4 - 1 - \sigma)! (j_1 + j_2 + j_3 + l_4 - 1 - \sigma)! (j_1 + j_2 + j_3 + l_4 - 1 - \sigma)!} \\
&\times \left( \begin{array}{ccc}
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\end{array} \right). \\
\end{align*}
\]

(16.2)

Here $\Delta$ is the triangle coefficient defined by (4.5) and $\sigma$ runs over all integral values which do not lead to negative arguments in the factorials. The sum (16.2) may be rewritten in the symmetrical and easily memorized form:

\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= \Delta(j_1 j_2 j_3) \Delta(j_1 j_3 j_4) \Delta(j_2 j_4 j_1) \Delta(j_3 j_1 j_2) \\
&\times \sum_{\sigma} (-1)^{j_1 + j_2 + j_3 + l_4 + \sigma} \left( \begin{array}{ccc}
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\end{array} \right) \\
&\times \frac{1}{(j_1 + j_2 + j_3 + l_4 - 1 - \sigma)! (j_1 + j_2 + j_3 + l_4 - 1 - \sigma)! (j_1 + j_2 + j_3 + l_4 - 1 - \sigma)!} \\
&\times \left( \begin{array}{ccc}
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\sigma & j_1 + j_2 + j_3 + l_4 - 1 - \sigma \\
\end{array} \right). \\
\end{align*}
\]

(16.2a)

Like the Clebsch-Gordan coefficient, the $6j$-coefficient may be expressed in terms of the hypergeometric function /Rose 1957/

\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= \frac{\Delta(j_1 j_2 j_3) \Delta(j_1 j_3 j_4) \Delta(j_2 j_4 j_1) \Delta(j_3 j_1 j_2) (-1)^{j_1 + j_2 + j_3}}{\left( j_1 + j_2 + j_3 - 1 \right)! (j_1 + j_2 + j_3 - 1)! (j_1 + j_2 + j_3 - 1)!} \\
&\times \left( \begin{array}{ccc}
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\end{array} \right) \\
&\times \frac{1}{(j_1 + j_2 + j_3 - l_4 - \sigma)! (j_1 + j_2 + j_3 - l_4 - \sigma)! (j_1 + j_2 + j_3 - l_4 - \sigma)!} \\
&\times \left( \begin{array}{ccc}
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\sigma & j_1 + j_2 + j_3 - l_4 - \sigma \\
\end{array} \right). \\
\end{align*}
\]

(16.3)

The triad structure of the $6j$-coefficient is quite evident from the definition (16.1) and particularly from the diagram in Figure 16.1. The parameters form four triads which are the arguments of the four triangle coefficients in (16.2) and (16.3).

The symmetry properties of the $6j$-coefficient are as follows:

\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= \{ j_3 j_1 j_2 \} = \{ j_2 j_3 j_1 \} = \{ j_3 j_2 j_1 \} \\
&\equiv \frac{1}{(a, b, c = 1, 2, 3)}. \\
\end{align*}
\]

(16.4)

(16.4) gives the 24 symmetry properties found by Racah (1942) and Wigner (1937). These properties are easily found from expression (16.2) or with the aid of the diagram in Figure 16.1. To illustrate the symmetries, consider, for example,

\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= \{ j_2 j_1 j_3 \} = \{ j_3 j_1 j_2 \} = \{ j_3 j_2 j_1 \}. \\
\end{align*}
\]

Evidently, this permutation changes nothing in (16.2), and in the diagram in Figure 16.1 it corresponds to a rotation of the figure about the line $j_3$ through $180^\circ$, with a subsequent change in signs of all the nodes and change in the directions of the lines $l_1$, $l_2$ and $l_3$. It is easily seen that these changes give the phase factor $+1$.

To obtain certain other properties in (16.4) from (16.2) one must replace the summation parameter $\sigma$ by $\sigma + (\text{combination of the parameters})$. To obtain them from the graphical representation in Figure 16.1 it is necessary to re-draw the figure in such a way that $j_3$ and $l_3$ become the sides of a quadrilateral. This is done by redistributing the nodes on the closed line (the so-called Hamilton line). A more elegant method of obtaining the 24 symmetry properties (16.4) would be to give the diagram of the $6j$-coefficient the form of a tetrahedron, the symmetry group of which will then give these properties /Edmonds 1957/.

The sum (16.2) can be simplified when one of the parameters is zero, or when the sum of two parameters in some triad is equal to the third parameter of this triad (in both cases the triangle of the parameters stretches into a line). In the first case we have

\[
\begin{align*}
\{ j_1 j_2 j_3 \} &= \{ j_1 j_2 l_4 \} = (-1)^{j_1 + j_2 + l_4} \frac{1}{(j_1 + j_2 + l_4)!} \\
&\times \left( \begin{array}{ccc}
j_1 & j_2 & l_4 \\
l_4 & j_1 & l_2 \\
l_4 & l_2 & j_1 \\
\end{array} \right). \\
\end{align*}
\]

(16.5)
In the second case
\[
\begin{align*}
\{j_1h_1l_1j_2h_2l_2\} &= (-1)^{h_1h_2l_1l_2} \times \\
\times &\left\{\frac{1}{4}(j_1 + j_2 + 1)(j_1 + j_2 - 1)(j_1 + j_2 - l_1 - l_2)\right\}^{\frac{1}{2}}.
\end{align*}
\]
(16.6)

These formulas simplify the calculation of certain 6j-coefficients.

The 6j-coefficient satisfies certain recurrence relations /Biedermann 1953/ which are obtained from (24.9) by assigning particular values to certain parameters.

We shall quote these relations. To increase a parameter by \(\frac{1}{2}\) /Obi et al. 1953, Edmonds 1957/:
\[
\begin{align*}
\{j_1h_1l_1\} &= \{(j_1 + 1)(j_1 + h_1 + 1)(j_1 + h_1 - 1)\} \times \\
\times &\left\{\frac{1}{4}(j_1 + h_1 + 1)(j_1 + h_1 - l_1)\right\}^{\frac{1}{2}} \\
\times &\left\{\frac{1}{4}(j_1 + h_1 + 1)(j_1 + h_1 - l_1)\right\}^{\frac{1}{2}}.
\end{align*}
\]
(16.7)

The 6j-coefficient also possesses the following property /Regge 1959/:
\[
\begin{align*}
\{j_1h_1l_1\} &= \left\{\frac{1}{4}(j_1 + h_1 + 1)(j_1 + h_1 - l_1)\right\}^{\frac{1}{2}} \\
\times &\left\{\frac{1}{4}(j_1 + h_1 + 1)(j_1 + h_1 - l_1)\right\}^{\frac{1}{2}}.
\end{align*}
\]
(16.9)

This relation enables us to replace one value of a parameter by another, whereas from (16.4) one can only change their position. (16.9) is therefore more of a functional relation than symmetry as is usually understood, as by symmetry is meant permutation of the arguments. For the case in which two columns in the 6j-coefficient are such that the sum of three parameters equals the fourth, this coefficient may be brought to the form (16.5) by using (16.9). Thus
\[
\begin{align*}
\{\frac{5}{2} \frac{1}{2} \frac{3}{2}\} &= \left\{\frac{5}{2} \frac{3}{2} \frac{1}{2}\right\}.
\end{align*}
\]

Let us list the available tables of 6j-coefficients. The tables of Simon et al. /1954/ (see appendix 1) give approximate values of \(W(j_1j_2l_1l_2j_3l_3)\) for
\[
\begin{align*}
\nu &= 0\left\{\frac{1}{2} \frac{3}{2} \frac{1}{2}\right\} \\
j_1 &= 0\left\{\frac{1}{2} \frac{1}{2} \frac{1}{2}\right\} \\
j_2 &= 0\left\{\frac{1}{2} \frac{1}{2} \frac{1}{2}\right\} \\
l_1 &= 0\left\{\frac{1}{2} \frac{1}{2} \frac{1}{2}\right\}
\end{align*}
\]
to nine places of decimals calculated with the aid of electronic computers. The tables of Obi et al. give exact values of \(W\) in the form of fractions for all integral values of the parameters within the limits
\[
\begin{align*}
j_1, j_2, l_1 &> l_2; j_2 > l_2 \\
l_1 + l_2 + l_3 &< 15; l_2, j_2 &= 0(1, 7)
\end{align*}
\]

/Obi et al. 1953/ and for integral \( f_4, l_4, l_5 \) and half-integral \( f_4, l_5, f_5 \) within the limits

\[
f_4 = l_4, j_4 = l_5 \quad \text{for} \quad j_4 = l_4;
\]

\[
l_4 + l_5 + l_6 \leq 15;
\]

\[
l_5 = 0(1) 7; \quad l_6 = \frac{1}{2} \left( \frac{1}{2} \right) 13;
\]

/Obi et al. 1954, 1955/.

Boys and Sahai /1954/ give tables of numerical values of the quantity \( U \) which is closely related to the 6-j-coefficient (see appendix 1) for the parameters

\[
j_1, j_2, j_3, l_4 = 0(1) 4 \quad \frac{5}{2};
\]

\[
j_5, l_4 = 0(1) 4 \quad \frac{5}{2};
\]

Briedenham /1952/ gives tables of the function \( W \) for

\[
j_4 = 0(1) 4; \quad l_4 = 0(1) 5; \quad j_5, l_4 = 0(1) 4 \frac{5}{2};
\]

\[
j_5 = \frac{1}{2}(2 \frac{5}{2}) 3; \quad l_5 = 0(1) 8.
\]

In some cases it is possible to use algebraic formulas for the 6-j-coefficient for certain values of one of the parameters. These formulas are easily obtained from the expression (16.2), which was transformed by Sato /1955/ to the following form:

\[
\begin{align*}
\{ j_1 + j_3, j_2, j_4, j_5 \} = & -(-)^{j_1-j_2+j_3+j_4+j_5} \\
\times & \left[ (j_3+j_5)(j_2+2j_4)(j_4+j_5)(j_1+j_4)^{(j_2+j_5)} (j_1+j_5)^{(j_2+j_4)} \right] \\
\times & \frac{(j_1+j_3)(j_2+j_3-j_4+j_5)(j_2+j_4+j_5)}{(j_2+j_3+j_4+j_5)^{(j_1+j_3)} (j_1+j_3+j_4+j_5)} X
\end{align*}
\]

(16.10)

Here

\[
\begin{align*}
\nu^{(V)} = & V(V-1) \cdots (V-A+1) \\
\nu^{(A+b)} = & V(V+1) \cdots (V+B) \\
A & = -j_1 + j_4 + l_4 \\
B & = j_1 - l_4 + l_6 \\
C & = j_3 + l_4 - l_5 \\
D & = j_1 + l_4 + l_5
\end{align*}
\]

(16.11)

Formula (16.10) has been written for the case \( j_4 \geq |j_4| \). If this condition is not satisfied, we obtain the required expressions for \( j_4 \geq |j_4| \) by changing \( j_4 \) to \( l_4 \) and \( j_5 \) to \( k_4 \). The last factor is expanded symbolically by the binomial theorem.

Algebraic formulas obtained from (16.10) for \( j_4 = 3, \frac{5}{2}, 4 \) and \( j_5 = 3, \frac{5}{2} \) are given in Obi et al. /1954, 1955/ and Sato /1955/.

They are given in a slightly different form for \( j_4 = 3, \frac{5}{2} \), 1, \( \frac{5}{2} \) and 2 by Briedenham et al. /1952/; and for \( j_4 = \frac{3}{2} \) by Edmonds and Flowers /1952/.

Asymptotic formulas for the 6-j-coefficient for large parameters may be found in Briedenham /1953/ and in Brunn and Tolhoek /1957/.

17. \( 3nj \)-coefficients of the first and second kinds

The 6-j-coefficient is the simplest nontrivial \( j \)-coefficient. In a certain sense it is possible to regard it as the principal \( j \)-coefficient, and to express all \( j \)-coefficients as sums of products of 6-j-coefficients. In this section we will consider certain single sums of products of 6-j-coefficients, which represent \( j \)-coefficients of the two types called \( 3nj \)-coefficients of the first and second kind /Levinson and Vanagas 1957/.

The sum of a product of 6-j-coefficients which represents a \( 3nj \)-coefficient of the first kind is of the form

\[
\begin{align*}
\{ j_1, j_2, \ldots, j_n \} = & \sum_{x} (-1)^{j_1 + j_2 + \cdots + j_n} \\
\times & \left[ j_1, j_2, \ldots, j_n \right] \left[ j_1, j_2, j_3 \ldots, j_n \right] \\
\times & \left[ j_1, j_2, j_3 \ldots, j_n \right] \left[ j_1, j_2, j_3 \ldots, j_n \right]
\end{align*}
\]

(17.1)

and the sum of products which gives a \( 3nj \)-coefficient of the second kind is

\[
\begin{align*}
\{ j_1, j_2, \ldots, j_n \} = & \sum_{x} (-1)^{j_1 + j_2 + \cdots + j_n} \\
\times & \left[ j_1, j_2, \ldots, j_n \right] \left[ j_1, j_2, \ldots, j_n \right] \\
\times & \left[ j_1, j_2, \ldots, j_n \right] \left[ j_1, j_2, \ldots, j_n \right]
\end{align*}
\]

(17.2)

Here

\[
R_n = \sum_{x=1}^{n} (j_1 + k_1 + l_1).
\]

(17.3)

In 3nj-coefficients of the first kind, triads are formed by the following sets of parameters: \( \{ j_1, j_2, j_3 \} \), \( \{ j_4, j_5, j_6 \} \), \( \{ j_7, j_8, j_9 \} \), \( \{ j_{10}, k_1, l_1 \} \), \( \{ j_{11}, k_2, l_2 \} \), \( \{ j_{12}, k_3, l_3 \} \), \( \{ j_{13}, k_4, l_4 \} \), \( \{ j_{14}, k_5, l_5 \} \), \( \{ j_{15}, k_6, l_6 \} \), \( \{ j_{16}, k_7, l_7 \} \), \( \{ j_{17}, k_8, l_8 \} \), \( \{ j_{18}, k_9, l_9 \} \), \( \{ j_{19}, k_{10}, l_{10} \} \), \( \{ j_{20}, k_{11}, l_{11} \} \), \( \{ j_{21}, k_{12}, l_{12} \} \), \( \{ j_{22}, k_{13}, l_{13} \} \), \( \{ j_{23}, k_{14}, l_{14} \} \). It is useful
to note that the expansions (17.1) and (17.2) differ only in the phase factor \((-1)^{\ell_1}\) and in the distribution of the parameters \(j_1\) and \(k_1\) in the last factor under the summation sign. This leads to different intersection properties of the lines \(j_1\) and \(k_1\) in the diagrams representing 3\(nj\)-coefficients of the first and second kind (Figures 17.1 and 17.2 respectively).

![Diagram](image)

Figure 17.1

Obviously, there need not be only one pair of intersecting lines. However, by deforming the diagram one can erase two pairs of intersecting lines without essentially changing the given sum. An uneven number of intersections therefore leads to Figure 17.1, and an even number to Figure 17.2. Hence it follows that, apart from those we have examined, no other forms of single sums exist.

The symmetry properties of 6\(j\)-coefficients (16.4) give the following symmetry properties for 3\(nj\)-coefficients of the first and second kinds respectively:

\[
\begin{pmatrix}
j_1 & j_2 & \ldots & j_n \\
l_1 & l_2 & \ldots & l_n \\
k_1 & k_2 & \ldots & k_n
\end{pmatrix}
= \begin{pmatrix}
k_1 & k_2 & \ldots & k_n \\
l_1 & l_2 & \ldots & l_n \\
j_1 & j_2 & \ldots & j_n
\end{pmatrix}
= \begin{pmatrix}
j_2 & j_3 & \ldots & j_n & j_1 \\
l_2 & l_3 & \ldots & l_n & l_1 \\
k_2 & k_3 & \ldots & k_n & k_1
\end{pmatrix},
\]

(17.4)

To obtain the first set of relations one must permute the rows in the 6\(j\)-coefficients in (17.1) and (17.2) in the columns which do not contain the summation parameter.

![Diagram](image)

Figure 17.2

To obtain the second set of relations the first 6\(j\)-coefficient must be transferred to the last position. Finally, to obtain the last of these relations one must write all the 6\(j\)-coefficients in reverse order and permute the rows and columns, without changing the columns containing the summation parameter.

It follows immediately from the triad structure that the sum of four parameters taken from two columns in the outer row is an integer.

If one of the parameters in the 3\(nj\)-coefficient vanishes then we obtain, taking
From the symmetry properties (17.4) and (17.5), the first row can be interchanged with the third; consequently, (17.7) and (17.9) also cover the case where the vanishing parameter is in the third row. Vanishing of the parameter in the middle row, as we see, leads to a 3 \((n-1)j\)-coefficient, i.e., reduces the number of parameters by three.

For \(n = 1\), from (17.1) and (17.2) we have the trivial expressions

\[
\begin{bmatrix} j_l \cr h_k \end{bmatrix} = (-1)^{l+k+h} \begin{bmatrix} j_l h_k \end{bmatrix},
\]

(17.10a)

Thus the 3 \(nj\)-coefficient of the first kind for \(n = 3\) is the well-known \(9j\)-coefficient, which is sometimes referred to as Fano's \(X\)-coefficient or Wigner's \(9j\)-coefficient.

If in (17.2) \(n = 3\), then we have

\[
\begin{bmatrix} j_l \cr h_k \end{bmatrix} = (-1)^{l+k-h} \begin{bmatrix} j_l h_k \end{bmatrix},
\]

For \(n = 2\)

\[
\begin{bmatrix} j_l \cr l_i \end{bmatrix} = (-1)^{l+k+h} \begin{bmatrix} j_l l_i \end{bmatrix},
\]

(17.11a)

\[
\begin{bmatrix} j_l \cr l_i \end{bmatrix} = (-1)^{l+k-h} \begin{bmatrix} j_l l_i \end{bmatrix},
\]

(17.11b)
The $f$-coefficients (18.2) and (18.3) are represented in Figures 18.1 and 18.2 respectively. It is therefore evident that the $9_f$-coefficient of the second kind (15.3) is not independent, since, as explained in section 14, diagram 18.2 is separable on the three lines $l_1$, $l_2$, and $l_3$. This gives

$$\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  l_1 & l_2 & l_3 \\
  k_1 & k_2 & k_3
\end{bmatrix} = (-1)^{l_1 + k_1} \begin{bmatrix}
  j_1 & j_2 & j_3 \\
  j_1 & j_2 & j_3 \\
  k_1 & k_2 & k_3
\end{bmatrix} \begin{bmatrix}
  l_1 & l_2 & l_3 \\
  l_1 & l_2 & l_3 \\
  k_1 & k_2 & k_3
\end{bmatrix} \begin{bmatrix}
  j_1 & j_2 & j_3 \\
  j_1 & j_2 & j_3 \\
  k_1 & k_2 & k_3
\end{bmatrix}.$$

(18.4)

The $9_f$-coefficient of the first kind (18.2), which is simply referred to as the $9_f$-coefficient, possesses a high degree of symmetry. Its symmetry properties may be written as follows /Wigner 1937, Jahn and Hope 1954/:

$$\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  l_1 & l_2 & l_3 \\
  k_1 & k_2 & k_3
\end{bmatrix} = (-1)^{l_1 + k_1} \begin{bmatrix}
  j_3 & j_2 & j_1 \\
  l_3 & l_2 & l_1 \\
  k_3 & k_2 & k_1
\end{bmatrix} \begin{bmatrix}
  j_3 & j_2 & j_1 \\
  l_3 & l_2 & l_1 \\
  k_3 & k_2 & k_1
\end{bmatrix} \begin{bmatrix}
  j_3 & j_2 & j_1 \\
  l_3 & l_2 & l_1 \\
  k_3 & k_2 & k_1
\end{bmatrix}.$$

(18.5)

They may be found, for instance, with the aid of diagram 18.1. This gives 72 symmetry properties in all. They show that the $9_f$-coefficient does not change upon transposition of the parameters relative to the two diagonals and upon an even permutation of the columns or rows, but that upon an odd permutation of the columns or rows it is multiplied by the phase factor $(-1)^{l_1 + k_1}$, where $R_3$ is the sum of all nine parameters. From the latter it follows that, when two rows or columns are equal, the following relation holds true:

$$\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  j_1 & j_2 & j_3 \\
  k_1 & k_2 & k_3
\end{bmatrix} = 0, \quad \text{if} \quad h_3 + h_2 + h_1 = \text{is odd},$$

(18.6)

It is readily seen that in the diagram of the $9_f$-coefficient it is possible to change the directions of all lines and the signs of all nodes without changing the phase factor. If one of the parameters of the $9_f$-coefficient vanishes, then from (18.2) and (18.5) it follows that

$$\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  l_1 & l_2 & l_3 \\
  h_1 & h_2 & h_3
\end{bmatrix} = 0, \quad \text{if} \quad h_1 + h_2 + h_3 = \text{is odd},$$

(18.7)

Since the $9_f$-coefficients are frequently encountered in calculations, we shall list the available tables of these coefficients. The tables of Smith and Stevenson /1957/ give values of $9_f$-coefficients to 10 places of decimals for values of the parameters

- $j_1 = \frac{1}{2} (1) \frac{3}{2}$; $j_2 = \frac{3}{2} (1) \frac{3}{2}$; $j_3 = \frac{3}{2} (1) \frac{3}{2}$

The supplement to these tables /Smith 1958/ gives

- $j_1 = \frac{3}{2} (1) \frac{3}{2}$; $j_2 = \frac{5}{2} (1) \frac{3}{2}$; $j_3 = \frac{3}{2} (1) \frac{3}{2}$

Tables of $9_f$-coefficients in fractional form have been computed by Sharp and his collaborators /Kennedy et al., 1954/ covering values of the parameters

1) $j_1 = h_2$, $j_3 = k_2 = \frac{1}{2}$, $j_4 = h_1 = 1, 2$.
2) $j_2 = h_1 = \frac{1}{2} (1) \frac{3}{2}$; $j_3 = k_2 = \frac{3}{2} (1) \frac{5}{2}$; $j_4 = k_1 = 3, 4$.
3) $j_3 = k_2 = 1 (1) 4$; $j_4 = \frac{1}{2}$; $k_2 = \frac{3}{2}$; $k_3 = \frac{5}{2}$.

Sharp et al. /1954/ give these coefficients for values of the parameters

1) $j_1 = h_2$, $j_3 = h_3 = \frac{1}{2} (1) 0, 1, 2, 3$; $j_4 = k_2 = 1, 2$.
2) $j_2 = h_1 = 1, h_2 = 2, h_3 = k_3 = h_2 = \frac{1}{2} (1) 5$.

Values of the quantity $A$ (see appendix 1), which differs from the $9_f$-coefficient by a simple factor, may be found in Kennedy and Cliff /1955/ in the form of decimals and fractions for

- $j_1, k_1 = 0 (1) 0, j_2, k_2 = 1, \frac{1}{2}$.

It may be found useful to refer to the tables of Matsumoto and Takebe /1955/, which contain algebraic formulas and numerical values in the form of fractions for the $9_f$-coefficient for $j_3 = k_3 = \frac{1}{2}, j_2, k_1 = 0 (1) 4$.
19. 12\textit{j}-coefficients and their properties

If in (17.1) and (17.2) \( n = 4 \), we obtain

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix} = \sum_{x} (x)(-1)^{n-x} \times
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix}
\times \begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix},
\]

(19.1)

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix} = \sum_{x} (x)(-1)^{n-x} \times
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix}
\times \begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix}.
\]

(19.2)

(19.1), which is a 12\textit{j}-coefficient of the first kind, was introduced by Jahn and Hope /1954/ and studied by Oed-Smith /1954/. It is usually referred to simply as the 12\textit{j}-coefficient.

It is convenient to use the following more symmetric quantity instead of (19.2):

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix} = (-1)^{h-k+k-h} \sum_{x} (x) \times
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix}
\times \begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix},
\]

(19.3)

which is related to (19.2) in the following way:

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix} = (-1)^{-h-k+h-j_1-j_3+j_2+j_4+h-k} \times
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix}
\times \begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix}.
\]

(19.4)

This quantity was introduced by Elliott and Flowen /1953/ and studied by Sharp /1953/ in a somewhat different notation, related to the present one /Vanagas and Chiplis 1958/ by

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 & j_4 \\
    h_1 & h_2 & h_3 & h_4
\end{bmatrix} = \begin{bmatrix} h_1 \cdot j_3 & j_2 \cdot j_1 \\ j_2 \cdot j_1 & h_1 \end{bmatrix}.
\]

(19.5)

We shall call this quantity the 12\textit{j}-coefficient of the second kind. It is convenient to write it as follows, in order to examine the triad structure:

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 \\
    h_1 & h_2 & h_3
\end{bmatrix}
\]

(19.6)

here the triads can readily be seen.

The diagrams which represent 12\textit{j}-coefficients on a plane are given in Figures 19.1a, 19.1b and 19.2. We see from these that neither the 12\textit{j}-coefficient of the first kind nor the 12\textit{j}-coefficient of the second kind can be simplified, as their diagrams cannot be separated on fewer than four lines.

It should be noted that a 12\textit{j}-coefficient of the first kind may be drawn on a plane as a polygon by the two methods indicated in Figures 19.1a and 19.1b, as a consequence of the symmetry properties which follow directly from (17.4). Hamilton's line, which should form the perimeter of a polygon when passing from diagram 19.1b to 19.1a, consists of the lines \( j_1, j_2, j_3, j_4, h_1, h_2, h_3, h_4 \).

Like the 3\textit{ji}-coefficient, the 12\textit{j}-coefficient of the second kind possesses symmetry properties in addition to those of the general 3\textit{ji}-coefficient.

These may be easily examined by representing the diagram of the 12\textit{j}-coefficient of the second kind in the form of a cube /Vanagas and Chiplis 1958/. The following symmetries are thus obtained:

\[
\begin{bmatrix}
    j_1 & j_2 & j_3 \\
    h_1 & h_2 & h_3
\end{bmatrix} = \begin{bmatrix} j_1 & j_3 & j_2 \\
    h_1 & h_3 & h_2
\end{bmatrix}.
\]

(19.7a)
\[
\begin{pmatrix}
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j
\end{pmatrix}
= \begin{pmatrix}
1_i \ 1_i \ 1_i \ 1_i \\
1_i \ 1_i \ 1_i \ 1_i \\
1_i \ 1_i \ 1_i \ 1_i \\
1_i \ 1_i \ 1_i \ 1_i
\end{pmatrix}
\begin{pmatrix}
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j
\end{pmatrix}
\begin{pmatrix}
1_i \ 1_i \ 1_i \ 1_i \\
1_i \ 1_i \ 1_i \ 1_i \\
1_i \ 1_i \ 1_i \ 1_i \\
1_i \ 1_i \ 1_i \ 1_i
\end{pmatrix}
\begin{pmatrix}
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j
\end{pmatrix}
\]
\[= \begin{pmatrix}
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j \\
1_j \ 1_j \ 1_j \ 1_j
\end{pmatrix}.
\]
it is easily seen that there exists in all only one possibility of joining the lines, shown in Figure 20.1 by the dotted lines. The diagram obtained gives the only possible 9j-coefficient, which admits of no further simplification.

Using either method, it is easily seen that for \( n = 4 \) there are only the two kinds of 12j-coefficients described in the preceding section. The number of different 15j-coefficients \( (n = 5) \) is 5. Their diagrams, symmetry properties and algebraic expansions /Levinson and Chiplis 1953/ are given below.

The diagrams of 15j-coefficients of the first, second, third, fourth and fifth kinds are given in Figures 20.2-20.6 respectively. As in the case of the 12j-coefficient of the first kind, representations of the diagrams of 15j-coefficients of the first, third and fourth kinds by plane polygons are not unique. For the sake of convenience, all such representations are given in the corresponding figures. As may be seen from Figure 20.6, the 15j-coefficient of the fifth kind cannot be represented by a plane polygon in which the vertices cover all the nodes of the given diagram.

The 15j-coefficients of the first and second kind are particular cases of the general coefficients examined in section 17. From (17.1) and (17.2), we obtain their expressions in terms of the sums of products of 6j-coefficients; their symmetry properties follow from (17.4) and (17.5). For the remaining types of 15j-coefficients, the symmetry properties are established with the aid of their diagrams. These coefficients are already incapable of expression as single sums of products of 6j-coefficients, although certain of these may be expressed as single sums in which 9j-coefficients appear in addition to 6j-coefficients. These expressions are obtained by the method of separating the diagrams given in section 14. We shall give these expressions and the symmetry properties for these three 15j-coefficients.
and is expressed as
\[ \left\{ \begin{array}{c} h_1, h'_1 \kappa \kappa' \kappa_2 \kappa_3 \\ j_1, j'_1 \p' \p_2 \p_3 \\ j' \end{array} \right\} = \sum_s (s) (-1)^{\p' + \p + \kappa - \kappa'} \times \sum_{j', j} \left\{ \begin{array}{c} h \times \kappa \kappa' \\ j \times j' \\ h_1 j_1 p_1 \p_2 \p_3 \end{array} \right\} \times \left\{ \begin{array}{c} h' \times \kappa' \\ j' \times j' \\ h_1 j_1 p_1 \p_2 \p_3 \end{array} \right\}. \] (20.3)

The symmetry properties of the 15j-coefficient of the fourth kind are
\[ \left\{ \begin{array}{c} j_1, j_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} = \sum_s (s) \left\{ \begin{array}{c} j_1, j_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} \times \left\{ \begin{array}{c} j_1, j_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\}, \] (20.4)

where
\[ \omega = h_1 - h_2 = \kappa_1 - \kappa_2 + 2s_1 + 2s_2. \] (20.5)

and it is given by
\[ \left\{ \begin{array}{c} j_1, j_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} = (-1)^{h_1 + h_2 - \omega + s_1 + s_2} \times \sum_s (s) \left\{ \begin{array}{c} j_1, j_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} \times \left\{ \begin{array}{c} j_1, j_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\}. \] (20.6)

Finally, for the 15j-coefficient of the fifth kind we have the symmetry properties
\[ \left\{ \begin{array}{c} h_1 h'_1 j_1 j'_1 \kappa_1 \kappa_2 \kappa_3 \\ h_2 h'_2 j_2 j'_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} = (-1)^{v_1} \left\{ \begin{array}{c} h_1 h'_1 j_1 j'_1 \kappa_1 \kappa_2 \kappa_3 \\ h_2 h'_2 j_2 j'_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} = (-1)^{v_2} \left\{ \begin{array}{c} h_1 h'_1 j_1 j'_1 \kappa_1 \kappa_2 \kappa_3 \\ h_2 h'_2 j_2 j'_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\} = (-1)^{v_3} \left\{ \begin{array}{c} h_1 h'_1 j_1 j'_1 \kappa_1 \kappa_2 \kappa_3 \\ h_2 h'_2 j_2 j'_2 \kappa_1 \kappa_2 \kappa_3 \\ j' \end{array} \right\}, \] (20.7)

where
\[ v_1 = l_1 + l_2 - l'_1 - l'_2, \]
\[ v_2 = l_1 + l_3 + l'_1 + l'_2 + l'_3 + j_1 + j_2 + j_3, \]
\[ v_3 = l_1 - l_2 + l'_2 - l'_3 + h_1 + h_2, \] (20.8)
and the expression

\[
\begin{pmatrix}
    h_1 & k_1' & j_1' \\
    h_2 & k_2' & j_2' \\
    h_3 & k_3' & j_3'
\end{pmatrix} = \sum_{x_1, x_2}(x_1)(x_2)\left[-1\right]^{l_1 + l_2 + k_1 + k_1' - k_2 - k_2' - l_3 + l_3'} \times
\]

\[
\begin{pmatrix}
    l_1 & l_2 & l_3 \\
    k_1 & k_2 & k_3 \\
    j_1 & j_2 & j_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
    l_1' & l_2' & l_3' \\
    k_1' & k_2' & k_3' \\
    j_1' & j_2' & j_3'
\end{pmatrix}.
\]

All the parameters in this 15j-coefficient are equivalent. The notation for the 15j-coefficient is taken from Levinson and Chiplis /1958/. It has been chosen for the clearest expression of the triad structure and symmetries.

Figure 20.6

The 15j-coefficient has been studied by Budzite and Gutman /1959/. Owing to the size of the diagrams and formulas, they have been relegated to appendices 3 and 4.

Chapter V

UTILIZATION OF TRANSFORMATION MATRICES FOR OBTAINING SUM RULES AND TRANSFORMATION FORMULAS FOR jm-coefficients

In sections 8 and 9 (Chapter II) we examined the general properties of transformation matrices of eigenfunctions of coupled angular momenta. It is evident from (8,8) that the elements of the transformation matrix can be expressed in terms of the product of two generalized Clebsch-Gordan coefficients summed over all the magnetic quantum numbers. As the generalized Clebsch-Gordan coefficients can be expressed in terms of generalized Wigner coefficients, we see that the elements of the transformation matrix may be expressed as sums of products of Wigner coefficients which are independent of the magnetic quantum numbers. In other words, these matrices may be expressed in terms of j-coefficients. Herein lies the main significance of j-coefficients, and the importance of studying the relation between transformation matrices and j-coefficients. By determining this relation one can obtain the properties of j-coefficients from the properties of transformation matrices. It is to this question that the present chapter is devoted.

Section 21 deals with the general consideration of the relation between transformation matrices and j-coefficients. In the next section we consider methods of separating the explicit expressions for the transformation matrix in terms of j-coefficients; section 23 gives these expressions for the case of the simplest transformation matrices. Section 24 is devoted to obtaining sum rules for products of j-coefficients from matrix identities. Finally, in section 25 we indicate a method for obtaining transformation formulas for jm-coefficients with the help of transformation matrices.

21. General considerations on the relation between transformation matrices and j-coefficients

When studying the relation between transformation matrices and j-coefficients, the transformation matrix may, without loss of generality, be characterized by a single permutation of the component angular momenta. In fact, the matrices

\[
(P_1 \cdots j_1)^a_j|\langle \cdots |P_2 \cdots j_2)^a_j|
\]

and

\[
(B_1 \cdots j_1)^a_j|\langle \cdots |B_2 \cdots j_2)^a_j|
\]

may be expressed in terms of j-coefficients of the same type. This follows from the fact that the matrix (21.2) may be obtained from (21.1) by renumbering the subscripts of the angular moments \(j_i\), which does not affect the structure of the matrix. We can
thus confine our attention to matrices of the following form:

\[
\begin{pmatrix}
(j_1 \ldots j_n) \cdot \sigma^a_f \cdot (P_{j_1} \ldots j_n) \cdot \sigma^a_f
\end{pmatrix}.
\]  

(21.3)

It is seen from here that for the given schemes of coupling \( A \) and \( B \) (see section 6), there can be \( n! \) matrices characterized by permutations of the subscripts \( 1, 2, \ldots, n \) in the right-hand side of (21.3). These matrices in which the same two angular momenta are added together on both sides need not be considered, since, owing to the unitarity of the Clebsch-Gordan coefficients, they may be expressed in terms of the products of transformation matrices of the eigenfunctions of \( n = 2 \) coupled angular momenta and the Kronecker delta. There exist several elementary ways of studying the remaining matrices; these make it possible to separate sets of matrices which may be expressed in terms of \( j \)-coefficients of the same type /Vanagas and Chipits 1958/. These methods are based on the symmetries of the Clebsch-Gordan coefficients (4.7c) and renumbering of the subscripts. We shall examine them in greater detail.

In accordance with (8.7), the matrix (21.3) and the matrix

\[
(P_{j_1} \ldots j_n) \cdot \sigma^a_f \cdot (P_{-1} \ldots j_n) \cdot \sigma^a_f
\]  

(21.4)

are equal. We shall renumber the subscripts \( j \) in the above matrix according to the permutation \( P^{-1} \). We then obtain the matrix

\[
(P_{j_1} \ldots j_n) \cdot \sigma^a_f \cdot (P_{-1} \ldots j_n) \cdot \sigma^a_f.
\]  

(21.5)

From the above, the matrices (21.4) and (21.5) are expressed in terms of \( j \)-coefficients of the same type. This property is useful when \( A \neq B \) and \( P \neq P^{-1} \).

If the scheme of coupling \( B \) in the matrix (21.3) is such that the angular momenta \( j_i \) and \( j_k \) are added directly and the matrix is characterized by the permutation

\[
P = \begin{pmatrix}
1 & 2 & \cdots & n
\end{pmatrix}_{j_1 \ldots i_k \ldots k \ldots i_1},
\]  

(21.6)

then it follows from the symmetry properties of the Clebsch-Gordan coefficients that the matrix characterized by the permutation

\[
P = \begin{pmatrix}
1 & 2 & \cdots & n
\end{pmatrix}_{j_1 \ldots i_k \ldots k \ldots i_1}
\]  

(21.7)

will differ from the first matrix only by a phase factor, i.e., both these matrices may be expressed in terms of \( j \)-coefficients of the same type.

If the coupling scheme \( A \) in the matrix (21.3) is such that the angular momenta \( j_i \) and \( j_f \) are added directly and the matrix is characterized by the permutation (21.6), then by interchanging these angular momenta in the left side of the matrix and renumbering the subscripts \( k \) and \( i \) so as to read \( f \) and \( t \) respectively, we will obtain a new matrix characterized by the permutation (21.7). Owing to the fact that a permutation of the coupled angular momenta can only change the phase of the matrix, and renumbering of the subscripts does not influence it structure, the original matrix and the one obtained can be expressed in terms of \( j \)-coefficients of the same type.

By using these elementary methods one can distinguish matrices which differ essentially from each other from those which do not. Thus, using the symmetry properties of the Clebsch-Gordan coefficients and renumbering the subscripts, one can bring any scheme of addition to a form in which the angular momenta \( j_i \) and \( j_f \) in the left-hand side and \( j_j \) and \( j_k \) in the right are added directly. Then the \( 2(n-2)! \) matrices for which \( j_1 = 1 \) and \( j_2 = 2 \) or \( j_2 = 2 \) and \( j_1 = 1 \) and \( j_4 = 1 \) may be expressed in terms of the product of the Kronecker delta and the transformation matrix of the eigenfunctions of \( n = 2 \) coupled angular momenta. Half of the remaining \( n! \cdot 2(n-2)! \) matrices have the same structure owing to the permutation of \( j_i \) and \( j_j \) (4.7c, 1, 2). Further, half the remaining matrices have the same structure owing to the possibility of renumbering the subscripts 1 and 2. Thus for any coupling schemes \( A \) and \( B \) it remains to consider not more than

\[
\frac{n! \cdot 2(n-2)!}{4}
\]  

(21.8)

matrices.

If the coupling schemes are such that more than one pair of angular momenta are directly added at a time, then the number of essentially different matrices is still further reduced. For example, it is easily seen that when the coupling scheme \( A \) is such that two pairs of angular momenta \( B \) are directly coupled, for any coupling scheme \( j_f \), it is sufficient to consider

\[
\frac{n! \cdot 6(n-2)!}{8}
\]  

(21.9)

matrices.

Let us consider in greater detail the influence of the coupling scheme on the structure of the transformation matrices. For a large number of added angular momenta \( n \), there exist many different schemes of coupling. In the simplest cases they are added in the following way. The angular momenta are divided into sets of, say, \( h_1, h_2, \ldots, h_\lambda \) angular momenta (2, \( \lambda = n \)). In each of these sets the angular momenta are added step-by-step, namely

\[
\ldots \left( \begin{pmatrix}
1 & + & +
\end{pmatrix} \left( \begin{pmatrix}
1 & + & +
\end{pmatrix} + \ldots \right) \right).
\]  

(21.10)

The resultant angular momenta of these sets are added in the same manner. This coupling scheme is conveniently denoted by the symbol \( A_{h_1 h_2 \cdots h_\lambda} \). Thus, for \( n = 6, h_1 = 2, h_2 = 3 \) and \( h_3 = 1 \), \( h_4 = h_5 = h_6 = 0 \), we have

\[
A_{2^{\frac{1}{2}} 3^{\frac{1}{2}}} = \left( \begin{pmatrix}
1 & + & +
\end{pmatrix} + \left( \begin{pmatrix}
1 & + & +
\end{pmatrix} + \ldots \right) \right).
\]  

(21.11)

For any scheme of coupling \( B \), \( A_{h_1 h_2 \cdots h_\lambda} \) and \( A_{h_1 h_2 \cdots h_\lambda} \) give matrices with identical structures. Consequently, the number of essentially different coupling schemes is substantially reduced. For small \( n \) this number is altogether insignificant. For \( n = 3 \) there is only one scheme \( A_{2} \); for \( n = 4 \) there are two, \( A_{2} \) and \( A_{2^{\frac{1}{2}}} \), and so forth.

Henceforth we shall use the following abbreviated notation for the various coupling schemes

\[
A_{2} = A_{2},
\]

\[
A_{2^{\frac{1}{2}}} = A_{2},
\]

\[
A_{2^{\frac{1}{2}}} = A_{2},
\]  

(21.12)
To simplify the writing we shall introduce a somewhat condensed notation for an element of the transformation matrix. Thus, the matrix (21.3) shall be denoted as follows:

\[(12 \ldots n)^a | (i_1 \ldots i_r)^b) = \sum_{a_1 \ldots a_n} \sum_{b_1 \ldots b_n} \delta^{a_1 a_2 \ldots a_n} (h_1 \ldots h_r)^b C \times (i_1 \ldots i_r)^b \cdot \delta^{b_1 b_2 \ldots b_n} = \sum_{a_1 \ldots a_n} \sum_{b_1 \ldots b_n} \delta^{a_1 a_2 \ldots a_n} (h_1 \ldots h_r)^b C \times (i_1 \ldots i_r)^b \cdot \delta^{b_1 b_2 \ldots b_n} \]  

(21.13)

In these symbols the component angular momenta will be indicated only by the corresponding subscripts. Intermediate and resultant angular momenta can be dropped.

From now on in this notation we shall write matrix identities of the type:

\[\sum_{a_1 \ldots a_n} \sum_{b_1 \ldots b_n} \delta^{a_1 a_2 \ldots a_n} (h_1 \ldots h_r)^b C \times (i_1 \ldots i_r)^b \cdot \delta^{b_1 b_2 \ldots b_n} = \sum_{a_1 \ldots a_n} \sum_{b_1 \ldots b_n} \delta^{a_1 a_2 \ldots a_n} (h_1 \ldots h_r)^b C \times (i_1 \ldots i_r)^b \cdot \delta^{b_1 b_2 \ldots b_n} \]  

(21.14)

in the abbreviated form

\[\sum_{a_1 \ldots a_n} \sum_{b_1 \ldots b_n} \delta^{a_1 a_2 \ldots a_n} (h_1 \ldots h_r)^b C \times (i_1 \ldots i_r)^b \cdot \delta^{b_1 b_2 \ldots b_n} = \sum_{a_1 \ldots a_n} \sum_{b_1 \ldots b_n} \delta^{a_1 a_2 \ldots a_n} (h_1 \ldots h_r)^b C \times (i_1 \ldots i_r)^b \cdot \delta^{b_1 b_2 \ldots b_n} \]  

(21.15)

Further, we shall denote the permutation (6, 17) by the symbol \(P_{14} \ldots 14\).

22. Methods for obtaining the relation between transformation matrices and \(f\)-coefficients

The expression for transformation matrices in terms of \(f\)-coefficients is most conveniently found by the graphical method. We therefore represent the generalized Wigner coefficients appearing in the expression for the transformation matrix in the right-hand side of (6.6) in graphical form according to the method given in section 12. Further summation over \(m_1, \ldots, m_n\) entails contracting the two diagrams; this leads to a closed diagram. For practical calculations it is useful to establish a general rule, given below, for expressing the transformation matrix in terms of \(f\)-coefficients which will enable us to obtain directly the phase factor as well.

The diagrams representing the generalized Wigner coefficients should be drawn in such a way that the nodes representing the triads on the right side of the matrix have a "+" sign; lines representing the compounded angular momenta should be directed towards the node, and the line representing the resultant should be directed away from the node. Further, the lines representing the compounded angular momenta in a triad should be so placed that the "first" can be brought into coincidence with the "second" by a counter-clockwise rotation. On the left side of the matrix, the nodes should have a "-" sign and the sense of the lines and the mutual position of the first and the second line in a triad should be the reverse of the above. After drawing the nodes and the lines in this fashion, all corresponding lines should be contracted. A direct calculation will show that the transformation matrix will be equal to the \(f\)-coefficient represented by the diagram set up in the manner indicated above, multiplied by

\[(-1)^{2(f + h_1 + h_2 + S)} C \times \prod_{i=1}^{\infty} (2a_i + 1)(2b_i + 1) \frac{1}{\sqrt{2}} \]  

(22.1)

where \(S\) is the sum of all "first" coupled angular momenta, and

\[C = \left[ \frac{1}{\prod_{i=1}^{\infty} (2a_i + 1)(2b_i + 1)} \right] \frac{1}{\sqrt{2}} \]  

(22.2)

For the coupling scheme \(A = B = A_2\), the phase factor may be rewritten as follows

\[(-1)^{2(f + h_1 + h_2 + \sum_{i=1}^{\infty} a_i)} \]  

(22.3)

where \(\sum_{a_i}\) is the sum of all intermediate angular moments on the left side of the matrix.

---

As an example of the use of the above rules, we quote the expression

\[\langle 12345 \rangle^{h_1}(13254)^{h_2}\]  

\[(-1)^{2(h_1 + h_2 + \sum_{i=1}^{\infty} a_i)} \times \left[ (J_{12}(J_{13})J_{12}(J_{13})J_{12}) \right]^{h_1} \times F \]  

(22.4)

where \(F\) is the \(f\)-coefficient represented by Figure 22.1a, which may be brought to
the form of Figure 22.1b. Further, comparing Figures 22.1b and 19.2, we obtain

\[
(12345)^{A_4|(13254)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

\[
(13254)^{A_4|(13245)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

\[
(13245)^{A_4|(13254)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

\[
(13254)^{A_4|(13245)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

\[
(13245)^{A_4|(13254)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

\[
(13254)^{A_4|(13245)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

\[
(13245)^{A_4|(13254)} = \left(\left( (i_1 i_2) (i_3 i_4) (i_5) \right) (i_3 i_4) (i_5) \right)
\]

Transformation matrices may also be studied by a purely algebraic method. For this it is necessary to reduce the matrix to a sum of products of simpler matrices with known expressions in terms of \( J \)-coefficients. The required expression is obtained by carrying out the summation. As an example, let us express the matrix under consideration as

\[
(12345)^{A_4|(13254)} = \sum_{A_4} \left( (12345)^{A_4|(132354)} (13254)^{A_4|(132354)} \right)
\]

All the matrices in the right-hand side may be expressed in terms of \( J \)-coefficients after performing the simplification (23.1), given in the following section. Subsequently, the expression (22.5) may be recovered by using (19.3).

It should be noted that the graphical method is the most convenient for studying the transformation matrices, as it does not involve a search for means of expressing the given matrix in terms of simpler matrices. Moreover, this expression is sometimes rather artificial in character and is not always easy to find.

23. Explicit expressions for the simplest transformation matrices

In this section we shall examine the expressions for transformation matrices for three, four and five coupled angular momenta. Of the matrices which are reducible to each other by elementary methods (as described in the preceding section) we choose the matrix which is characterized by the highest permutation in the lexicographic order.

As we noted in section 21, for three angular momenta there is only the scheme of coupling \( A_6 \). It follows from (21.8) that in this case it is sufficient to consider the matrix characterized by the permutation \( P_{134} \). This matrix was examined for the first time by Racah /1943/. It is expressed in terms of the \( J \)-coefficients as follows:

\[
(123)^{A_6|(132)} = \left( -1 \right)^{i_1 + i_2} [J_{13} J_{13}] \left( \begin{array}{c} i_1 i_2 J_{13} \\ i_1 J_{13} \end{array} \right)
\]

For the addition of four angular momenta, there are two different schemes \( A_4 \) and \( A_6 \). One should therefore examine the matrices characterized by the schemes \( (A_4 | A_8), (A_8 | A_6) \), and \( (A_4 | A_6) \). It follows from (21.8) that there are five matrices

for the schemes \( (A_4 | A_8) \). The permutations which characterize these matrices are:

\[ P_{134}, P_{143}, P_{134}, P_{134}, P_{134}, \] and \( P_{134} \). The first three may be simplified by the method of section 9. The matrix \( P_{134} \) may be reduced to \( P_{134} \) by renumbering the subscripts. With this aid the expansion

\[
(1234)^{A_4|(1324)} = \left( (1234)^{A_4|(1324)} (1324)^{A_4|(1324)} \right)
\]

in which summation over the intermediate angular momenta drops out owing to diagonalization (see the end of section 8), the latter matrix may be reduced to the product of two \( J \)-coefficients. The matrix characterized by the permutation \( P_{134} \) is expanded in a similar way

\[
(1234)^{A_4|(3142)} = \left( (1234)^{A_4|(3142)} (3142)^{A_4|(3142)} \right)
\]

and therefore also reduces to the product of two \( J \)-coefficients. The matrix \( P_{134} \) can be expressed in terms of the \( J \)-coefficients as follows /Arima et al., 1954/.

\[
(1234)^{A_4|(1432)} = \left( -1 \right)^{i_1 + i_2} [J_{13} J_{13}] \left( \begin{array}{c} i_1 J_{13} J_{13} \\ i_1 J_{13} J_{13} \end{array} \right)
\]

\[
(1234)^{A_4|(1324)} = \left( -1 \right)^{i_1 + i_2} [J_{13} J_{13}] \left( \begin{array}{c} i_1 J_{13} J_{13} \\ i_1 J_{13} J_{13} \end{array} \right)
\]

It follows from (23.1) that for the coupling schemes \( (A_4 | A_8) \) it is sufficient to consider the two matrices \( P_{134} \) and \( P_{134} \). The two are expanded as follows:

\[
(1234)^{A_4|(1324)} = \left( (1234)^{A_4|(1324)} (1324)^{A_4|(1324)} \right)
\]

and

\[
(1234)^{A_4|(1432)} = \left( (1234)^{A_4|(1432)} (1432)^{A_4|(1432)} \right)
\]

The first expansion was obtained by Biedenharn /1953/. These show that both matrices can be expressed as products of \( J \)-coefficients.

For the coupling schemes \( (A_4 | A_8) \) it is sufficient to study one of the matrices which cannot be simplified further, e.g., \( P_{134} \). It was examined by Wigner and an expression for it given by Jahn and Hope /1954/.

It is

\[
(1234)^{A_4|(1324)} = \left( (1234)^{A_4|(1324)} J_{13} J_{13} \right)
\]

\[
(1234)^{A_4|(1324)} = \left( (1234)^{A_4|(1324)} J_{13} J_{13} \right)
\]

Thus, for four added angular momenta, only two types of all possible transformation matrices require to be expressed in terms of \( J \)-coefficients, while the rest reduce to the product of two \( J \)-coefficients.

Let us turn to the transformation matrices of eigenfunctions of five coupled angular momenta. Here the possible coupling schemes are \( A_4, A_6, A_6 \). Consequently, the transformation matrices of five coupled angular momenta can be characterized by the following schemes:

\( (A_4 | A_6), (A_6 | A_6), (A_6 | A_6), (A_6 | A_6), (A_6 | A_6) \) and \( (A_6 | A_6) \). An analysis similar to the one given above for four angular momenta leads to the results which are given in Table 23.1. The table lists the coupling schemes and the permutations (upon the order 1234). Of the matrices
which are reducible to each other, only the lexicographically highest one is shown in
the table. Thus, to obtain the matrix \( (A_{i} | A_{j}) \) with the permutation \( P \), one
must permute the angular momenta \( (j_{1}, j_{2}) \) and \( (j_{i}, j_{i}) \) interchange the subscripts
1 and 2, make use of the reality of the matrix and again renumber the subscripts 4,
5, 1, 2, 1 to 2, 3, 4, 5 respectively. One then obtains the matrix with the permuta-
tion \( P_{i} \), which, according to the table, can be expressed as a product of
6j- and 9j-coefficients.

Explicit expressions as products of 6j- and 9j-coefficients may be found for
the matrices of the first three groups in this table with the aid of the expansions
\( A_{1}(5), A_{2}(5) \) in appendix 5, which also gives expressions for the remaining matrices
in terms of 12j-coefficients \( (A_{1}(5), A_{2}(5)) \).

In addition to the special matrices considered above, two general transformation
matrices are known to be expressed in terms of 3nj-coefficients. The matrix
which can be expressed in terms of the 3nj-coefficient of the first kind is /Levitt
and Vanagas 1957/

\[
\begin{align*}
\left( 1 \ldots \ldots h n a - 1 \ldots \ldots h + 1 \right)^{b^{e}} & = (1 \ldots \ldots h n a - 1 \ldots \ldots h + 1)^{b^{e}} \\
(2 \ldots \ldots h n b - 1 \ldots \ldots h + 1)^{b^{e}} & = (2 \ldots \ldots h n b - 1 \ldots \ldots h + 1)^{b^{e}} \\
\end{align*}
\]

\[
\begin{align*}
\times \left( J_{1} \ldots \ldots J_{n} \right) \times \left( J_{1} \ldots \ldots J_{n} \right) \\
\times \left( J_{1} \ldots \ldots J_{n} \right) \\
(23,8)
\end{align*}
\]

If here \( h = n - 1 \), then \( A_{1} \) becomes
\( (12 \ldots \ldots n)^{a^{i}}(02 \ldots \ldots n - 1)^{b^{i}} = (02 \ldots \ldots n - 1)^{a^{i}}(12 \ldots \ldots n - 1)^{b^{i}} \\
\times \left( J_{1} \ldots \ldots J_{n} \right) \times \left( J_{1} \ldots \ldots J_{n} \right) \\
\times \left( J_{1} \ldots \ldots J_{n} \right) \\
(23,9)
\]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( A_{i} | A_{j} \) & \( A_{i} | A_{j} \) & \( A_{i} | A_{j} \) & \( A_{i} | A_{j} \) & \( A_{i} | A_{j} \) & \( A_{i} | A_{j} \) \\
\hline
13254 & 35412 & 13254 & 35412 & 13254 & 35412 \\
13452 & 35142 & 13452 & 35142 & 13452 & 35142 \\
13524 & 34152 & 13524 & 34152 & 13524 & 34152 \\
34152 & 13524 & 34152 & 13524 & 34152 & 13524 \\
34512 & 15234 & 34512 & 15234 & 34512 & 15234 \\
43512 & 15324 & 43512 & 15324 & 43512 & 15324 \\
13542 & 13452 & 13542 & 13452 & 13542 & 13452 \\
14352 & 13542 & 14352 & 13542 & 14352 & 13542 \\
15342 & 15432 & 15342 & 15432 & 15342 & 15432 \\
35142 & 15432 & 35142 & 15432 & 35142 & 15432 \\
\hline
\end{tabular}
\caption{Summary of results on transformation matrices of five coupled angular moments}
\end{table}

The 3nj-coefficient of the second kind is related to the following matrix:
\( (12 \ldots \ldots n)^{a^{i}}(145 \ldots \ldots n 23)^{b^{i}} = (145 \ldots \ldots n 23)^{b^{i}} \\
\times \left( J_{1} \ldots \ldots J_{n} \right) \times \left( J_{1} \ldots \ldots J_{n} \right) \\
\times \left( J_{1} \ldots \ldots J_{n} \right) \\
(23,10)
\]

These formulas will be utilized later on.
24. Utilization of matrix identities for obtaining sum rules on \( f \)-coefficients

The connection between matrices and \( f \)-coefficients makes it possible to obtain sum rules on \( f \)-coefficients from matrix identities. To obtain the actual formulas one must first find the corresponding matrix identities and substitute \( f \)-coefficients for the matrix elements.

The simplest sum rules on the \( f \)-coefficients, so-called orthogonality relations, are obtained from the unitarity conditions of the transformation matrices. For example, using (23.9) we obtain

\[
\sum_{n_1, \ldots, n_n} \left\langle \hat{n}_1 \hat{n}_2 \ldots \hat{n}_n \right| \left( \hat{j}_1 \hat{j}_2 \ldots \hat{j}_n \right) = \delta_{(j_1, j_2, \ldots, j_n)} \delta_{(n_1, n_2, \ldots, n_n)} = \delta(j_1, j_2, \ldots, j_n) \delta(n_1, n_2, \ldots, n_n),
\]

where for the sake of convenience the angular momenta have not been written in the same way as in the original equation. For \( n=2 \), using (17.11a) we obtain from the above

\[
\sum_{n_1, n_2} \left\langle j_1 \quad n_1 \right| j_1 j_1 \left( \hat{j}_1 \hat{n}_1 \right) = \delta(j_1, j_1) \delta(n_1, n_1).
\]

This formula was first obtained by Racah [192.]

We note that there exist orthogonality relations for \( f \)-coefficients which may be expressed as products of simpler coefficients. Using the unitarity condition for the matrix (23.8) and formula (23.1), we obtain the following relation:

\[
\sum_{n_1, n_2} \left\langle n_1 \quad n_2 \right| \left( \hat{n}_1 \hat{n}_1 \right) = \delta(n_1, n_1) \delta(n_2, n_2).
\]

This formula expresses the \( (n-1) \)-fold sum of the product of two \( 3nj \)-coefficients as a single \( 3nj \)-coefficient. For \( n=2 \), using (17.11a) we obtain from (24.6) the following sum rule for \( 6f \)-coefficients [Racah 1942/]

\[
\sum_{n_1, n_2} \left\langle j_1 \quad j_1 \right| \left( \hat{j}_1 \hat{n}_1 \right) = \delta(j_1, j_1) \delta(n_1, n_1).
\]

However, this formula will not give a new sum rule for the \( Gf \)-coefficients, as it may be obtained by applying (24.2) twice.

Making use of the matrix identity

\[
\left\langle 12 \ldots n-1 \right| (n2 \ldots n-1) = 0
\]

we obtain the following sum rule for \( 3nj \)-coefficients of the first kind:

\[
\left( -1 \right)^{j_1-k_1-k_2-k_3-k_4} \sum_{n_1, \ldots, n_n} \left\langle n_1 \ldots n_n \right| \left( j_1 \ldots j_n \right) = \delta(j_1, j_2, \ldots, j_n) \delta(n_1, n_2, \ldots, n_n),
\]

(24.6)

(24.8)
If here \( n = 4 \) and \( f_k = 0 \), then, taking (17.7) into account, we obtain the following useful sum rule for \( 6j \)-coefficients: *Biedenharn 1953*

\[
\sum_{x} (-1)^{h_x} x \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ l_x & l_x & l_x \\ x & x & x \end{array} \right\} \cdot \left\{ \begin{array}{ccc} j_1 & j_4 & j_5 \\ l_1 & l_1 & l_1 \\ x & x & x \end{array} \right\} = \sum_{i=1}^{3} (j_i + h_i + l_i). 
\]

(24.9)

The important sum rules for \( j \)-coefficients are those which make it possible to express the sum of a product of \( j \)-coefficients as products of \( j \)-coefficients. A typical example of such a sum rule is

\[
(-1)^{h_a - h_{a-1} - h_b} \sum_{a_{a-1}} (-1)^{h_a - h_{a-1}} \left\{ \begin{array}{ccc} k_a & k_{a+1} & \cdots \\ l_a & l_{a+1} & \cdots \\ x_a & x_{a+1} & \cdots \\
 j_b & j_{b+1} & \cdots \\ l_b & l_{b+1} & \cdots \\ x_b & x_{b+1} & \cdots \\
j_a & j_{a+1} & \cdots \\ l_a & l_{a+1} & \cdots \\ x_a & x_{a+1} & \cdots \end{array} \right\} \times
\]

\[
\times \left\{ \begin{array}{ccc} k_a & k_{a+1} & \cdots \\ l_a & l_{a+1} & \cdots \\ x_a & x_{a+1} & \cdots \\
 j_b & j_{b+1} & \cdots \\ l_b & l_{b+1} & \cdots \\ x_b & x_{b+1} & \cdots \\
j_a & j_{a+1} & \cdots \\ l_a & l_{a+1} & \cdots \\ x_a & x_{a+1} & \cdots \end{array} \right\} = \left\{ \begin{array}{ccc} j_1 & j_2 & \cdots \\ l_1 & l_2 & \cdots \\ x_1 & x_2 & \cdots \\
j_b & j_{b+1} & \cdots \\ l_b & l_{b+1} & \cdots \\ x_b & x_{b+1} & \cdots \\
j_a & j_{a+1} & \cdots \\ l_a & l_{a+1} & \cdots \\ x_a & x_{a+1} & \cdots \end{array} \right\} \]

(24.10)

This formula may be obtained from the corresponding matrix identity or by multiplying (24.5) by

\[
\left\{ \begin{array}{ccc} j_1 & j_2 & \cdots \\ l_1 & l_2 & \cdots \\ x_1 & x_2 & \cdots \\
j_b & j_{b+1} & \cdots \\ l_b & l_{b+1} & \cdots \\ x_b & x_{b+1} & \cdots \\
j_a & j_{a+1} & \cdots \\ l_a & l_{a+1} & \cdots \\ x_a & x_{a+1} & \cdots \end{array} \right\}
\]

and summing over \( j_2 \cdots j_b \). This is the number of parameters in the first \( 3j \)-coefficient below the summation sign is not particularized.

We might mention yet another type of sum rule, which expresses a \( 3j \)-coefficient of the second kind as a single sum of a \( 3 \) \( (n + 1) \)-coefficient of the first kind. Using (17.2) and the particular case of (24.7) with \( h_b = 0 \), we obtain

\[
\left[ \begin{array}{ccc} j_1 & \cdots & j_a \\ l_1 & \cdots & l_a \\ h_1 & \cdots & h_a \end{array} \right] = \sum_{x} (-1)^{h_x} j_1 \left[ \begin{array}{ccc} j_1 & \cdots & j_a \\ l_1 & \cdots & l_a \\ h_1 & \cdots & h_a \end{array} \right].
\]

(24.11)

The formula for a \( 3j \)-coefficient of the first kind as a sum of a \( 3 \) \( (n + 1) \)-coefficient of the second kind is easily obtained in a similar way.

In this section we confined ourselves to certain characteristic sum rules for \( j \)-coefficients. By combining these or by using various matrix identities and the connection between matrices and \( j \)-coefficients, one can obtain a variety of different sum rules. The most important of these are given in appendix 6, which for convenience also lists the sum rules for \( 6j \)-, \( 9j \)- and \( 12j \)-coefficients given in this and in other sections.

25. Use of matrix identities for the transformation of \( jm \)-coefficients

Transformation matrices can be used to obtain formulas for the transformation of \( jm \)-coefficients. The derivation of these formulas is based on the relation (8.6), which expresses an element of the transformation matrix as a sum of products of generalized Clebsch-Gordan coefficients. Owing to the unitarity of the matrix of generalized Clebsch-Gordan coefficients, we can write this relation as

\[
\left( \begin{array}{c} j_1 m_1 \cdots j_a m_a (j_1 \cdots j_a)^2 a F M \end{array} \right) = \sum_{g} \left( \begin{array}{c} j_1 m_1 \cdots j_a m_a (j_1 \cdots j_a)^2 g F M \end{array} \right) \times \left( \begin{array}{c} j_1 \cdots j_a m_1 (j_1 \cdots j_a)^2 a g \end{array} \right),
\]

(25.1)

Relations between \( jm \)-coefficients can be obtained from this formula, together with the formulas for the expansion of transformation matrices as sums of products, provided the transformation matrices and Clebsch-Gordan coefficients are replaced by \( j \)-coefficients and Wigner coefficients respectively. Obviously, the structure of these formulas will also depend on the schemes of coupling of the angular momenta. We shall consider some examples of such formulas.

Let the transformation matrix which appears in (25.1) be similar to (23.9). In this case (introducing a more convenient notation for the parameters) one readily obtains from (25.1) the following formula for the transformation of \( jm \)-coefficients:

\[
\sum_{m_n \cdots m_1} (-1)^{m - m_n} \left( \begin{array}{ccc} j_1 & l_1 & j_a \\ m_1 & n_1 & m_a \end{array} \right) \left( \begin{array}{ccc} j_2 & l_2 & j_a \\ m_2 & n_2 & m_a \end{array} \right) \cdots \left( \begin{array}{ccc} j_{a-1} & l_{a-1} & j_a \\ m_{a-1} & n_{a-1} & m_a \end{array} \right)
\]

\[
\times \left( \begin{array}{ccc} j_a & l_a & k_a \\ -m_a & n_a & k_a \end{array} \right) = \sum_{m_n \cdots m_1} (-1)^{m + k + g} \left( \begin{array}{ccc} j_1 & l_1 & k_1 \\ m_1 & n_1 & g_1 \end{array} \right) \cdots \left( \begin{array}{ccc} j_a & l_a & k_a \\ m_a & n_a & g_a \end{array} \right)
\]

(25.2)

\[
\times \left( \begin{array}{ccc} j_1 & l_1 & k_1 \\ g_1 & n_1 & g_1 \end{array} \right) \cdots \left( \begin{array}{ccc} j_a & l_a & k_a \\ g_a & n_a & g_a \end{array} \right)
\]

(25.2)
Multiplying the above by

\((-1)^{n_2-n_1+m_1+m_2}\left(\begin{array}{ccc} k_1 & l_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right)\times \left(\begin{array}{ccc} k_2 & l_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) \times \left(\begin{array}{ccc} k_3 & l_3 & j_3 \\ m_3 & n_3 & m_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & l_4 & j_4 \\ m_4 & n_4 & m_4 \end{array}\right)\right)

summing over \(n_2\) and \(m_1\) and using the orthogonality property of the Wigner coefficients, we obtain the following sum rule for \(jm\)-coefficients:

\[
\sum_{n_2, m_1} \left(-1\right)^{n_2-n_1+m_1+m_2} \left(\begin{array}{ccc} k_1 & l_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right) \times \left(\begin{array}{ccc} k_2 & l_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) \times \left(\begin{array}{ccc} k_3 & l_3 & j_3 \\ m_3 & n_3 & m_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & l_4 & j_4 \\ m_4 & n_4 & m_4 \end{array}\right) =
\sum_{n_2, m_1} \left(\begin{array}{ccc} k_1 & l_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right) \times \left(\begin{array}{ccc} k_2 & l_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) \times \left(\begin{array}{ccc} k_3 & l_3 & j_3 \\ m_3 & n_3 & m_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & l_4 & j_4 \\ m_4 & n_4 & m_4 \end{array}\right). 
\]

For \(n = 2\), taking (17.11a) into account, we obtain a frequently encountered transformation formula

\[
\sum_{n_2, m_1} \left(-1\right)^{n_2-n_1+m_1+m_2} \left(\begin{array}{ccc} j_1 & j_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) = \left(-1\right)^{n_2+n_1+m_1+m_2} \left(\begin{array}{ccc} j_1 & j_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right). 
\]

which will be obtained in section 26 as an illustration of the graphical method.

For \(n = 3\) we have, from (25.3) and (18.1),

\[
\sum_{n_2, m_1} \left(-1\right)^{n_2-n_1+m_1+m_2} \left(\begin{array}{ccc} j_1 & j_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) \times \left(\begin{array}{ccc} j_3 & j_3 & j_3 \\ m_3 & n_3 & m_3 \end{array}\right) =
\sum_{n_2, m_1} \left(\begin{array}{ccc} j_1 & j_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) \times \left(\begin{array}{ccc} j_3 & j_3 & j_3 \\ m_3 & n_3 & m_3 \end{array}\right) \times \left(\begin{array}{ccc} j_4 & j_4 & j_4 \\ m_4 & n_4 & m_4 \end{array}\right) =
\sum_{n_2, m_1} \left(-1\right)^{n_2-n_1+m_1+m_2} \left(\begin{array}{ccc} j_1 & j_1 & j_1 \\ m_1 & n_1 & m_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_2 & j_2 \\ m_2 & n_2 & m_2 \end{array}\right) \times \left(\begin{array}{ccc} j_3 & j_3 & j_3 \\ m_3 & n_3 & m_3 \end{array}\right) \times \left(\begin{array}{ccc} j_4 & j_4 & j_4 \\ m_4 & n_4 & m_4 \end{array}\right).
\]

Using the symmetry properties of the \(9j\)-coefficient and (18.2), we can express the right-hand side of the above equation as follows:

\[
\sum_{k_1, k_2, k_3, k_4, k_5} \left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{array}\right) \left(\begin{array}{ccc} k_4 & k_5 & k_6 \\ j_4 & j_5 & j_6 \end{array}\right) = \left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & k_5 & k_6 \\ j_4 & j_5 & j_6 \end{array}\right)
\]

Transforming the sum over \(k_1\) with the aid of (25.2) for \(n = 2\), we obtain

\[
\sum_{k_1, k_2, k_3, k_4, k_5} \left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & k_5 & k_6 \\ j_4 & j_5 & j_6 \end{array}\right) =
\sum_{k_1, k_2, k_3, k_4, k_5} \left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & k_5 & k_6 \\ j_4 & j_5 & j_6 \end{array}\right) =
\sum_{k_1, k_2, k_3, k_4, k_5} \left(\begin{array}{ccc} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{array}\right) \times \left(\begin{array}{ccc} k_4 & k_5 & k_6 \\ j_4 & j_5 & j_6 \end{array}\right),
\]

The \(jm\)-coefficients considered above possess certain symmetry properties which can easily be examined by the graphical method.

Special notations are sometimes used for a few frequently encountered \(jm\)-coefficients. For example, Simon (1965) uses the following expression as an independent quantity

\[
G_m \left(\begin{array}{ccc} j_1 & j_2 & j_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_1 & j_1 \end{array}\right) = \left(-1\right)^{j_3+j_4+j_5} \left(\begin{array}{ccc} j_1 & j_2 & j_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_1 & j_1 \end{array}\right).
\]

which possesses the symmetry

\[
G_m \left(\begin{array}{ccc} j_1 & j_2 & j_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_1 & j_1 \end{array}\right) = \left(-1\right)^{j_3+j_4+j_5} \left(\begin{array}{ccc} j_1 & j_2 & j_1 \end{array}\right) \times \left(\begin{array}{ccc} j_2 & j_1 & j_1 \end{array}\right),
\]

where \(\varphi=j_1+j_2+j_3+j_4+j_5+j_6\). The quantity \(G_m\) depends on eight angular
momentum parameters. It is therefore often called the $8j$-coefficient. This name, however, is not entirely successful as $G_{ij}$ is essentially a $jm$-coefficient in which certain magnetic quantum numbers are zero.

The examples of the method of obtaining transformation formulas for $jm$-coefficients given in this section were intended merely as illustrations. In practical calculations one meets with highly varied sums of products of Wigner coefficients and in the majority of cases the corresponding calculations are more conveniently carried out by the graphical method which was developed in Chapter III. Examples of such calculations will be given in the following chapter.

Chapter VI

EXAMPLES OF APPLICATION OF THE GRAPHICAL METHOD

In practical calculations both the summation of products of Wigner coefficients and $j$-coefficients and the simplification of these sums are of importance. In the two preceding sections we have indicated how this may be done with the help of matrix algebra. In the present chapter we shall deal with the use of the graphical method of this purpose. As the graphical method is convenient for carrying out the summations, we considered it advisable to give a number of examples of the practical application of this method.

In section 26 we give the general method of graphical summation of a product of Wigner coefficients over magnetic quantum numbers, together with simple examples. In the next section we analyze a more complex case. Section 28 is devoted to the summation of products of $j$-coefficients. In section 29 we consider an example of summation of a product of $j$-coefficients and a $jm$-coefficient. At the end of the section is sketched a practical prescription for the simultaneous evaluation of a multiple sum over angular momentum parameters.

Finally, section 30 deals with the question of choosing a method of calculation. The expediency of using two different methods of calculation is indicated and simple examples are given in which the graphical method is used to perform the summation and the phase factor is determined algebraically.

26. Graphical summation of products of Wigner coefficients

In this section we examine sums of products of Wigner coefficients in which $j$-coefficients do not appear explicitly. As we know, these sums arise from sums of products of Clebsch-Gordan coefficients, which do not contain transformation matrices in an explicit form. Our purpose in this section is to clarify the graphical method of summation by means of examples.

If the sum under consideration is a $j$-coefficient, i.e., is summed over all magnetic quantum numbers, then in order to express the summation one must draw the corresponding diagram according to the method discussed in section 12. If the diagram obtained is not separable on less than four lines, it must be compared, after being brought to the form of a polygon, with the (conventionally) standard diagrams for the given number of parameters. The case of six or nine parameters will give the diagrams 16.1 or 18.1. Reduction of the directions of lines and of the signs (using the rules in section 12) to those in the above diagrams will give the phase factor of the required expression. For 12 parameters we will obtain one of the two diagrams in Figure 15.1 (for a $12j$-coefficient of the first kind) or the diagram in Figure 19.2.
(for a 12 \(j\)-coefficient of the second kind). For 15 parameters the diagram obtained will be identical with one of the diagrams in section 20. The diagrams of 18 \(j\)-coefficients in appendix 3 give only one possibility for each 18 \(j\)-coefficient, without going into all possible forms. Consequently, if a diagram obtained fails to coincide with any of these diagrams, it should be re-drawn in another form.

Should one obtain a diagram separable on less than four lines, it should first be separated by the method indicated in section 14; the separate parts should then be identified in the manner indicated above.

When the sum under consideration is a \(jm\)-coefficient, one must make use of theorem (13.5) for the expansion of a \(jm\)-coefficient in generalized Wigner coefficients. When using this theorem for actual calculations, one should draw a diagram representing the sum under consideration, choose a generalized Wigner coefficient and contract the two diagrams. The directions of free lines should be chosen appropriately in advance. The diagram obtained will represent a \(j\)-coefficient which is identified in the above manner and which will be the expansion coefficient in (13.5). To illustrate the method, we shall consider simple instances of these sums.

Let us first consider the sum

\[
F(\mathbf{l}_1 \mathbf{m}_1, \mathbf{j}_1 \mathbf{m}_1, \mathbf{l}_2 \mathbf{m}_2, \mathbf{j}_2 \mathbf{m}_2) = \sum_{\mathbf{n}_1 \mathbf{n}_2} (-1)^{\mathbf{l}_1 \mathbf{n}_2 + \mathbf{j}_1 \mathbf{n}_1 + \mathbf{l}_2 \mathbf{n}_2 + \mathbf{j}_2 \mathbf{n}_1} \times
\]

\[
\left( \begin{array}{c} l_1 j_1 l_2 j_2 \\ n_1 m_1 n_2 m_2 \end{array} \right) \left( \begin{array}{c} l_2 j_2 l_1 j_1 \\ -n_2 m_2 n_1 m_1 \end{array} \right) \left( \begin{array}{c} j_1 \ j_1 \ j_2 \ j_2 \\ n_1 m_1 n_2 m_2 \end{array} \right).
\]

(26.1)

The diagram of this \(jm\)-coefficient (Figure 26.1) is the simplest nontrivial \(jm\)-coefficient (the trivial case would have been the usual Wigner coefficient) with three free lines. In the given case the chosen generalized Wigner coefficient is a usual Wigner coefficient with the angular momenta \(j_1, j_2, j_3\), the graphical representation of which shows the lines issuing from a node. Changing the directions of the lines in the diagram of the Wigner coefficient, contracting it with the diagram of the \(jm\)-coefficient and comparing the diagram obtained with Figure 15.1, we obtain the following formula

\[
\sum_{\mathbf{n}_1 \mathbf{n}_2} (-1)^{\mathbf{l}_1 \mathbf{n}_2 + \mathbf{j}_1 \mathbf{n}_1 + \mathbf{l}_2 \mathbf{n}_2 + \mathbf{j}_2 \mathbf{n}_1} \times
\]

\[
\left( \begin{array}{c} l_1 j_1 l_2 j_2 \\ n_1 m_1 n_2 m_2 \end{array} \right) \left( \begin{array}{c} l_2 j_2 l_1 j_1 \\ -n_2 m_2 n_1 m_1 \end{array} \right) \left( \begin{array}{c} j_1 \ j_1 \ j_2 \ j_2 \\ n_1 m_1 n_2 m_2 \end{array} \right) =
\]

\[
(-1)^{l_1 + j_1 + j_2} \left( \begin{array}{c} l_1 j_1 l_2 j_2 \\ n_1 m_1 n_2 m_2 \end{array} \right) \left( \begin{array}{c} j_1 \ j_1 \ j_2 \ j_2 \\ m_1 m_2 m_1 m_2 \end{array} \right),
\]

(26.2)

which is a somewhat different form of (25.4).

Let us now analyze in detail the sum (11.5), the graphical representation of which is given in Figure 26.2a. It is the simplest \(jm\)-coefficient with four free lines,

![Diagram F](image)

![Diagram V](image)

Figure 26.2a

provided we disregard the trivial case of the generalized Wigner coefficient with four angular momenta (three component and one resultant).

Let us choose the generalized Wigner coefficient given by diagram \(V\) in Figure 26.2a. Joining the lines \(j_1, j_2, j_3\) and \(j_4\) in this diagram to the corresponding lines in diagram \(F\) (which retain their directions), we obtain diagram \(R\), which

![Diagram R](image)

![Diagram R'](image)

Figure 26.2b

can be represented as \(R'\) (Figure 26.2b). Comparing \(R'\) with the diagram of the
9j-coefficient in Figure 18.1, we obtain

\[ \mathbf{R} = \mathbf{R}' = (-1)^{l_1 + l_3 - l_2 - l_4} \begin{pmatrix} l_3 l_2 l_1 \\ l_2 l_3 l_1 \\ l_1 l_3 l_2 \end{pmatrix} \]  

(26.3)

Thus from (13.5) we finally have

\[ \sum_{n_3,n_4} (-1)^{l_1 + l_3 - l_2 - l_4} \times \begin{pmatrix} l_1 l_2 l_3 \\ -n_3 m_3 n_4 \\ -n_3 m_4 n_2 \end{pmatrix} \times \begin{pmatrix} l_2 l_3 l_4 \\ -l_2 n_3 m_4 n_2 \end{pmatrix} \times \begin{pmatrix} l_4 l_2 l_3 \\ -n_4 m_3 n_2 \end{pmatrix} = \sum_{a} (a) (-1)^{a \cdot \mathbf{b}} = \begin{pmatrix} l_1 l_2 l_3 \\ l_4 l_2 l_3 \\ j_1 j_2 j_3 \end{pmatrix} \]

\[ \times \begin{pmatrix} l_1 l_2 l_3 \\ l_4 l_2 l_3 \\ j_1 j_2 j_3 \end{pmatrix} \times \begin{pmatrix} a j_1 j_2 j_3 \\ m_1 m_2 m_3 \end{pmatrix} \times \begin{pmatrix} a j_1 j_2 j_3 \\ -\mu m_1 m_2 m_3 \end{pmatrix} \]  

(26.4)

Another advantage of the general formula (13.5) is that by changing the appearance of the generalized Wigner coefficient in it one can change the j-coefficient \( \mathbf{R} \).

Figure 26.3a

choosing it in the simplest or most convenient form. Thus by interchanging the parameters \( j_3 \) and \( j_4 \) in the generalized Wigner coefficient in our example, we obtain the \( j \)-coefficient \( \mathbf{R} \) in the form represented in Figure 26.3a. Let us re-draw it in the form \( \mathbf{R}' \), change the direction of the line \( l_4 \) and separate \( \mathbf{R}' \) on \( a, l_2, l_4 \), using (14.5). We then find

\[ \mathbf{R} = \mathbf{R}' = (-1)^{l_2 + l_4 - l_1 - l_3} \begin{pmatrix} l_3 l_2 l_1 \\ l_2 l_3 l_1 \\ l_1 l_3 l_2 \end{pmatrix} \]  

(26.5)

\( \mathbf{R}_1 \) and \( \mathbf{R}_2 \) are shown in Figure 26.3b. Comparing the diagrams of \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \) with the diagram of the 6j-coefficient in Figure 16.1, we obtain finally

\[ F(\mathbf{R}_1 ; \mathbf{R}_2) = \sum_{a} (a) (-1)^{a \cdot \mathbf{b}} = \begin{pmatrix} l_1 l_2 l_3 \\ j_1 j_2 j_3 \end{pmatrix} \times \begin{pmatrix} l_1 l_2 l_3 \\ j_1 j_2 j_3 \end{pmatrix} \times \begin{pmatrix} m_1 m_2 m_3 \end{pmatrix} \times \begin{pmatrix} -\mu m_1 m_2 m_3 \end{pmatrix} \]  

(26.6)

This last expression is a different form of the right-hand side of (26.4). Obtaining these formulas by the graphical method is considerably simpler than the derivations given earlier in section 25.

27. A more complex product of Wigner coefficients

In the preceding section we discussed simple examples of the summation of products of Wigner coefficients. In this section we will examine a more complex example in which there are five magnetic quantum numbers which are not summed over.

Consider the following sum of a product of seven Clebsch-Gordan coefficients

\[ \sum_{a} (-1)^{a} \begin{pmatrix} k_1 \cdot m_1 & k_2 \cdot m_2 & k_3 \cdot m_3 \\ k_1' \cdot m_1' & k_2' \cdot m_2' & k_3' \cdot m_3' \end{pmatrix} \times \begin{pmatrix} l_1 \cdot m_1 & l_2 \cdot m_2 & l_3 \cdot m_3 \\ l_1' \cdot m_1' & l_2' \cdot m_2' & l_3' \cdot m_3' \end{pmatrix} \times \begin{pmatrix} j_1 \cdot m_1 & j_2 \cdot m_2 & j_3 \cdot m_3 \\ j_1' \cdot m_1' & j_2' \cdot m_2' & j_3' \cdot m_3' \end{pmatrix} \times \begin{pmatrix} a & j_1 j_2 j_3 \\ m_1 m_2 m_3 \end{pmatrix} \times \begin{pmatrix} a & j_1 j_2 j_3 \\ -\mu m_1 m_2 m_3 \end{pmatrix} \]  

(27.1)

(the summation is over \( m_1, m_2, m_3, m_1', m_2', m_3', j_1, j_2, j_3 \) which occur in Simons 1953). The sum (27.1) is somewhat different from the sum (2.3) in the latter work, as it is written in the form which would result if one always placed the complex conjugate functions in the left-hand side of the matrix element in the calculation.

Further, we have retained all the summations and have not particularized the values of \( m_1 \) and \( m_2 \) which are usually zero, and of \( m_3 \) which is then \( q' = q \). If we express the Clebsch-Gordan coefficients in terms of Wigner coefficients with the aid of (5.2), then, apart from a trivial factor, the sum (27.1) will be reduced to the following \( jm \)-coefficient:

\[ C = \sum_{a} (-1)^{a} \begin{pmatrix} k_1 + k_2 + \cdots + k_3 + k_4 + \cdots + k_7 \cdot m_1 \cdot m_2 \cdot m_3 \\ -q - m_1 m_2 \end{pmatrix} \times \begin{pmatrix} l_1 + l_2 + \cdots + l_3 + l_4 + \cdots + l_7 \cdot m_1 \cdot m_2 \cdot m_3 \\ -\mu - m_1 m_2 \end{pmatrix} \times \begin{pmatrix} j_1 + j_2 + \cdots + j_3 + j_4 + \cdots + j_7 \cdot m_1 \cdot m_2 \cdot m_3 \\ -\mu j_1 j_2 j_3 \end{pmatrix} \]  

(27.2)
The diagram of this \(jm\)-coefficient is given in Figure 27.1.

Let us now expand the \(jm\)-coefficient \(C\) in generalized Wigner coefficients with the help of (13,5), choosing for the generalized Wigner coefficient the expression \(V\) corresponding to the diagram shown in Figure 27.1. We have

\[
C = \sum_L (L) V (L) R V.
\]  
(27.3)

To obtain the diagram of the \(j\)-coefficient \(R\), one must join the lines \(l_1, l_2, k, k', l'\) in diagrams \(C\) and \(V\), with the directions as in diagram \(C\). The diagram \(R\) thus obtained is shown in Figure 27.2. This diagram is separable on the three lines \(J_0, L, J_0\). The \(j\)-coefficient \(R\) therefore decomposes into the product of two \(j\)-coefficients

![Figure 27.1](image1)

![Figure 27.2](image2)

with fewer parameters. So as to make use of (14,5) we change the direction of the line \(J_0\). We then obtain the new \(j\)-coefficient \(R'\), where

\[
R' = (-i)^{2j} R.
\]  
(27.4)

From (14,5) we have

\[
R' = X_1 \cdot X_2.
\]  
(27.5)

The diagrams \(X_1\) and \(X_2\) (Figure 27.3) are not separable on fewer than four lines. They correspond to \(3j\)-coefficients. This will readily be seen by re-drawing, say, diagram \(X_2\) in the form of a hexagon (Figure 27.4). Comparing this diagram with diagram 18,1, we find

\[
X_2 = (-1)^{\psi_2} \begin{pmatrix} J_1 & \bar{J}_1 & \bar{J}_1 \\ L & k & l' \\ J_0 & \bar{J}_0 & \bar{J}_0 \end{pmatrix}.
\]  
(27.6)

where

\[
\psi_2 = i_1 + i_2 + l + 2J_1 + 2J_2 + 2L + 2J_1 + 2L + 2J_2.
\]  
(27.6a)

Analogously we obtain

\[
X_1 = (-1)^{\psi_1} \begin{pmatrix} J_1 & \bar{J}_1 \\ L & k \end{pmatrix}.
\]  
(27.7)

where

\[
\psi_1 = 2J_1 + 2J_2 + 2L + 2J_2.
\]  
(27.7a)

The algebraic expression for the generalized Wigner coefficient \(V\) is

\[
V = \sum_{\mu\nu M} (-1)^{\mu + \nu + M} \begin{pmatrix} L & \bar{L} \\ -p_1 & p_2 \end{pmatrix} \begin{pmatrix} M & \bar{M} \\ \bar{p}_1 & p_2 \end{pmatrix} \times \begin{pmatrix} L & \bar{L} \\ -p_1 & p_2 \end{pmatrix}.
\]  
(27.8)

Here summation is actually limited to the single term in which \(\mu = p_2 - p_1\) and
\[ M = q - q' = q + p_1 - p_2. \]

Introducing all the expressions obtained into (27,3), we obtain, for \( p_1 = p_2 = 0, \)
\[
C_{l_h, l_k, n_h, n_k} = (-1)^{h_h + h_k - l_h - l_k} \sum_{l_i} \left( \begin{array}{ccc} l_i & l_i & l_i \\ L & k & L \end{array} \right) \left( \begin{array}{ccc} l_i & l_i & l_i \\ L' & k' & L' \end{array} \right) \times \\
\times \left( \begin{array}{ccc} l_h & l_h & l_h \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_k & l_k & l_k \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_2' & l_2' & l_2' \\ -q' & -q' \end{array} \right).
\] (27,9)

A comparison with (25,6) shows that in (27,9) we have, apart from a trivial factor, the so-called \( 8f \)-coefficient (see the end of section 25).

28. Summation of a product of \( f \)-coefficients.

In practical calculations one frequently encounters sums of products of \( f \)-coefficient over one or several parameters. The result of this summation is again a \( f \)-coefficient, the diagram of which may be derived from the rules in section 15. In certain instances this diagram may be separable on fewer then four lines; the \( f \)-coefficient one obtains will then decompose into the product of simpler \( f \)-coefficients.

To calculate the \( f \)-coefficients obtained upon summation, they may conveniently be expressed in terms of sums of products of \( f \)-coefficients with fewer parameters. Various expressions of this kind can be obtained by the method of separation elaborated in section 14. For numerical calculations it is preferable to have sums of the least possible multiplicity, containing \( f \)-coefficients with the smallest possible number of parameters. Such formulas are given for \( 9f \)-, \( 12f \)- and \( 18f \)-coefficients in terms of \( 6f \)- and \( 9f \)-coefficients in sections 17-20 and in appendix 6. As extensive tables are available only for the \( 6f \)-coefficients, the \( 9f \)-coefficients entering into the expressions under consideration may be expressed in terms of the latter, although this will increase the multiplicity of the sum.

If the original sum is \( n \)-fold and the resulting diagram is \( n \)-separable on less than \( n + 3 \) lines, the multiplicity of the sum cannot be reduced. However, by determining and then separating the diagram which represents such a sum, it may be possible to express the sum in different forms and sometimes even to simplify it. We shall illustrate this with a simple example.

It is easy to verify the following identity \( \text{Vilibrantze and Jacys 1959/} \) for the \( 12f \)-coefficient of the second kind:
\[
\left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 \end{array} \right) = (-1)^{h_h - h_k + k_h} \sum_x \left( \begin{array}{ccc} l_1 & l_1 & x \\ h_1 & h_1 & h_1 \\ k_1 & k_1 & k_1 \end{array} \right) \left( \begin{array}{ccc} l_2 & l_2 & x \\ h_2 & h_2 & k_2 \\ j_1 & j_1 & j_1 \end{array} \right) \left( \begin{array}{ccc} l_3 & l_3 & x \\ h_3 & h_3 & j_3 \\ k_3 & k_3 & j_3 \end{array} \right).
\] (28.1)

The graphical representation of this \( 12f \)-coefficient is given in Figure 19. Assume that we are trying to evaluate the sum in the right-hand side of (28.1) and have found its diagram. It is separable only on four \((1 + 3)\) lines. It would therefore seem that the sum cannot be simplified. If, however, after separation on the lines \( l_1, l_2, l_3 \)

and \( l_4 \) one joins the line \( l_1 \) to \( l_4 \) and \( l_2 \) to \( l_4 \), then the two resulting parts will be separable on three lines. This reduces to a single sum of a product of \( 4f \)-coefficients (formula 19.3). The multiplicity of the sum has not changed, but the summed has been simplified: Instead of the product of two \( 9f \)-coefficients (each of which, from (18.2), is a summed product of three \( 6f \)-coefficients), we have a product of \( 6f \)-coefficients.

In most cases summation results in a marked simplification of the expression under consideration. As an example of this, consider the following formula \( \text{Atima et al. 1954/} \)
\[
\sum_{xy} (-1)^{x+y+1} \left( \begin{array}{ccc} k_h & k_h & k_h \\ j_k & j_k & j_k \end{array} \right) \left( \begin{array}{ccc} h_x & h_x & h_x \\ y_k & y_k & y_k \end{array} \right) \left( \begin{array}{ccc} l_x & l_x & l_x \\ h_y & h_y & h_y \end{array} \right) \left( \begin{array}{ccc} j_x & j_x & j_x \\ k_y & k_y & k_y \end{array} \right) =
\]
\[
(-1)^{d + h + j - l} \left( \begin{array}{ccc} f_x & f_x & f_x \\ h_k & h_k & h_k \end{array} \right) \left( \begin{array}{ccc} l_l & l_l & l_l \\ j_h & j_h & j_h \end{array} \right).
\] (28.2)

which expresses a double sum as a single product of \( 6f \)- and \( 9f \)-coefficients. We shall illustrate the graphical summation of \( f \)-coefficients with this characteristic example.

Figure 28.1

Let us denote the required sum by \( K \) and note that from the "multiplication rule" of section 15 the product of the three \( 6f \)-coefficients in (28.2) may be represented by the diagram \( K_3 \) in Figure 28.1. \( K \) can then be written as
\[
K = \sum_{xy} (-1)^{x+y+1} \langle x,y \rangle K_1 K_2
\] (28.3)
where $K_j$ is the $j$-coefficient in Figure 28.1. We change the directions of the lines $K_1$ and $j$ in diagram $Y_j$, and obtain

$$K_j = (-1)^{j_1 + j_2} K'_j = (-1)^{j_1 + j_2 + j_3} K'_j,$$

(28.4)

where $K'_j$ is the $j$-coefficient represented by the diagram $K_j$ after the directions of these two lines have been changed. Using the multiplication rule for the product $K'_1 K'_2$, we obtain the diagram $K_3 = K'_1 K'_2$. The original sum then becomes

$$K = \sum_n \delta (\gamma) (-1)^{j_1 + j_2 + j_3} K'_j.$$

(28.5)

We change the sign of one of the nodes of the line $Y_j$ so as to be able to use the "summation rule" of section 15. This gives

$$K = (-1)^{j_1 + j_2 + j_3} \sum_n \delta (\gamma) K'_j.$$

(28.6)

The summation is carried out by dropping the lines $X$ and $Y_j$ from diagram $K'_j$ together with the adjacent nodes and joining corresponding lines; as a result we obtain

![Diagram X1 and X2](image)

Figure 28.2

the diagram $X_1$ in Figure 28.2. Re-drawing it as $X_2$, and changing the direction of the line $l_1$, we find

$$X_1 = X_2 = (-1)^{2j} X_1.'$$

(28.7)

Separating $X_1'$ on the lines $l_1, l_2$ and $l_3, l_4, l_5$ we obtain the two diagrams $Y_1$ and $Y_2$ in

![Diagram Y1 and Y2](image)

Figure 28.3

we must first change the direction of $k_3$, we find

$$X = (-1)^{2j} Y_1 Y_2.$$

(28.12)
Comparing the diagrams \( Y_1 \) and \( Y_2 \) with the diagrams in Figure 16.1, we find
\[
Y_1 = (-1)^{l_1 + l_2 - k_1 - k_2} \begin{cases} j_1 l_1 l_2 \\ k_1 k_2 k_3 \end{cases}, \quad (28,13)
\]
\[
Y_2 = (-1)^{l_1 + k_1 + k_2} \begin{cases} j_1 j_2 j_3 \\ k_1 k_2 k_3 \end{cases}. \quad (28,14)
\]
From (28,11)-(28,14) we finally obtain
\[
\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  k_1 & k_2 & k_3 \\
  0 & 0 & 0
\end{bmatrix} = (-1)^{l_1 + l_2 + l_3} \frac{1}{(l_1)(l_3)} \begin{bmatrix}
  j_2 & j_3 & j_1 \\
  k_2 & k_3 & k_1 \\
  l_2 & l_3 & l_1
\end{bmatrix}, \quad (28,15)
\]
which is essentially identical with (19,10).

29. Summation of a product of Wigner coefficients and \( f \)-coefficients

In this section we will give an example of the graphical summation of a product in which both Wigner coefficients and \( f \)-coefficients appear explicitly. The summation may extend over the angular momentum and the magnetic quantum numbers. Consider the sum
\[
H = \sum_{n=\text{all}} (-1)^{n + r + s + t} P(n) \left( \begin{array}{c}
  l_1 \\
  n_1
\end{array} \right) \left( \begin{array}{c}
  l_2 \\
  n_2
\end{array} \right) \left( \begin{array}{c}
  k_1 \\
  n_3
\end{array} \right) \times
\]
\[
\begin{bmatrix}
  j_1 & j_2 & j_3 \\
  s & s & s
\end{bmatrix} \left( \begin{array}{c}
  r \\
  s
\end{array} \right) \left( \begin{array}{c}
  k_1 & k_2 & k_3 \\
  r & r & r
\end{array} \right) \left( \begin{array}{c}
  j_1 & j_2 & j_3 \\
  s & s & s
\end{array} \right). \quad (29,1)
\]
Summation over the magnetic quantum number \( q \) gives the \( jm \)-coefficient \( V \) (the generalized Wigner coefficient of Figure 29.1). The required sum may then be written as
\[
H = \sum_{n=\text{all}} (-1)^{n + r + s + t} P(n) \left( \begin{array}{c}
  l_1 \\
  n_1
\end{array} \right) \left( \begin{array}{c}
  l_2 \\
  n_2
\end{array} \right) \left( \begin{array}{c}
  k_1 \\
  n_3
\end{array} \right) V W X_1 X_2. \quad (29,2)
\]
The individual factors in the above are represented in the diagrams in Figure 29.1, where for convenience identical nodes have been indicated by (the same) numbers.

Let us first sum over \( k \), applying the rule for the summation of a product to \( V X_1 X_2 \). In order to do this, we will have to change the directions of the lines \( n k, r k, l_1 k \) and \( k_1 k \) in the diagram \( X_2 \). This will bring us to the diagram \( X_3 \). Now,
\[
X_3 = (-1)^{2n + 2s + 2t} X_2, \quad \text{since } 2n + 2s + 2t \text{ is an even number.}
\]
We have
\[
H = \sum_{n=\text{all}} (-1)^{n + r + s + t} P(n) (-1)^{2n + 2s + 2t} W \sum_{k=\text{all}} (k) V X_1 X_2 = \sum_{n=\text{all}} (-1)^{n + r + s + t} P(n) \left( \begin{array}{c}
  l_1 \\
  n_1
\end{array} \right) \left( \begin{array}{c}
  l_2 \\
  n_2
\end{array} \right) \left( \begin{array}{c}
  k_1 \\
  n_3
\end{array} \right) W A. \quad (29,3)
\]
The diagram of the \( jm \)-coefficient \( A \) is obtained from the rule for the summation of a product. Separating the lines adjoining the line \( k \) in diagrams \( V, X_1 \) and \( X_2 \) and

\[\text{Figure 29.1}\]

\[\text{Figure 29.2}\]

\[\text{Figure 29.3}\]

\*

In certain cases, if the lines are labelled by angular momenta alone, there may be more than one pair of free lines with the same index in the separated diagrams. An additional label would then be required to define "corresponding lines" uniquely, and this is the number of the node to which the line converged before separation of the diagrams. It would be seen that lines which correspond "non-significantly" converged to different nodes before separation.

\*

98

99
those parts of the diagrams which contain these lines. In this example we are then left with the diagrams in Figure 29.7. Joining the ends of corresponding lines converging to the same nodes, we obtain the diagram of the sum. For our example, this immediately gives the diagram C. In the general case, the validity of the rule for obtaining the diagram of a sum is evident if one can assume that when the summation is performed step by step, after the summation over each parameter it is possible to further sum over one of the remaining parameters with the aid of our rules. For this to hold true the original expression to be summed must satisfy certain conditions, which, however, have not yet been established. All sums encountered in practice satisfy this condition. Presumably it is related to the requirement that the result of the summation be a $jm$-coefficient.

30. Choice of a method of calculation

In sections 26–29, it was shown how various transformations of sums of $j$-coefficients and Wigner coefficients are carried out by the graphical method. These transformations may also be carried out algebraically. This requires the use of identities such as the ones given in appendices 6 and 7, which are usually obtained from matrix identities as described in sections 24 and 25.

Each method possesses certain advantages and disadvantages. The algebraic method is convenient only when one has a sufficiently complete list of sum rules and transformation formulas for $j$- and $jm$-coefficients. The main drawback of this method is its lack of generality: for each concrete case one must choose the appropriate transformations leading to the required result, often a very difficult task. It should be noted that once the method of transformation is known, the use of algebraic formulas gives the final result rather rapidly.

When the graphical method is used, $j$ and $m$ are summed over with the help of two standard rules which are applicable in all cases without exception. The number of elementary operations is somewhat larger than when algebraic formulas are used. The result of the summation is a $jm$-coefficient obtained in the form of a diagram. The method of transformation is easy to guess owing to the clarity of the graphical operations.

In certain cases it is convenient to combine the two methods. Use of non-oriented diagrams makes it possible to establish the result very rapidly up to the phase factor. If one can thus discern the necessary algebraic transformations, the phase factor can be found by carrying them out. Usually, however, this is possible only in cases which are not very complicated. Using non-oriented diagrams, one can foresee the usefulness of algebraic transformations, while it is usually impossible to tell by inspection whether or not an algebraic sum can be evaluated.

As an illustration of the use of both methods in conjunction, let us consider the following example:

$$ P = \sum x \langle j_2 j_1 l_3 | l_2 x k_3 \rangle \langle j_2 j_3 l_1 | k_2 k_1 k_2 \rangle. \tag{30.1} $$

We draw the $6j$-coefficient and the $9j$-coefficient with the lines of the summed parameters lying close to each other (Figure 30.1), but without paying attention to the directions of the lines or to the signs of the nodes. We drop the parameter $x$ and the nodes adjacent to it, and join corresponding lines, as indicated in Figure 30.1 by the dotted line. We re-draw this diagram as in Figure 30.2. This last diagram is separable on the three lines $j_3$, $k_1$, and $j$. Further, again disregarding the directions of lines and signs of nodes, we separate it into two parts and obtain two $6j$-coefficients (Figure 30.3). We then write

$$ P = (-1)^{l_2} \langle j_2 j_1 k_2 | j_3 j_2 k_3 \rangle \langle j_2 j_3 k_1 | k_3 k_2, \tag{30.2} $$

where $\psi$ is to be determined algebraically.
Chapter VII

IRREDUCIBLE TENSOR OPERATORS AND EXPRESSIONS
FOR THEIR MATRIX ELEMENTS

One of the principal applications of the mathematical apparatus discussed in the preceding chapters is the calculation of matrix elements of operators. This calculation may be simplified by the use of irreducible tensor operators. The corresponding methods have been developed by Wigner /1931/ and Racah /1942/, and it is in fact in their application that the mathematical apparatus described earlier was evolved. Use of these methods substantially simplifies the calculation of matrix elements.

The present chapter briefly discusses methods of calculating the matrix elements of irreducible tensor operators. In section 31 we consider general aspects of this method. The following section is devoted to the products of irreducible tensor operators. Section 33 gives expressions for the matrix elements of tensor products of two irreducible tensor operators. The next section gives an example of calculation of matrix elements of the tensor product of four irreducible tensor operators. The last section deals with irreducible double tensor operators and their products.

31. Irreducible tensor operators and their properties

The question of decomposing tensor operators to an irreducible set is closely related to the reduction of the tensor representation of the three-dimensional rotation group. The $3^r$ components of the tensor $T_1^{(r)}$ of rank $r$ form the basis of the tensor representation $\mathcal{T}$, which is the direct product of the vector representations $\mathcal{V}$ of the rotation group

$$\mathcal{T} = \mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V} \quad (r \text{ times}).$$

Decomposing the representation $\mathcal{T}$, we obtain

$$\mathcal{T} = \sum_{k} a_{k} \mathcal{D}_{k}.$$  \hspace{1cm} (31.2)

Here $a_{k}$ indicates the number of times the irreducible representation $\mathcal{D}_{k}$ appears in the decomposition of $\mathcal{T}$. We have the obvious equality

$$\sum_{k} a_{k} (2k + 1) = 3^r. \hspace{1cm} (31.3)$$

The decomposition of $\mathcal{T}$ amounts to transformation to a new basis consisting of the quantities $T^{(r)}_k$ ($k = -r, \ldots, r$) which are linear combinations of the quantities $T_1^{(r)}$. The set of $(2k + 1)$ quantities $T^{(r)}_k$ with fixed $k$ forms the basis of the irreducible representation $\mathcal{D}_{k}$ of the rotation group. This set is the irreducible tensor $T^{(r)}_k$. 

of rank \( k \) with components \( T^k_{\ell} \). The transformation properties of an irreducible tensor can be expressed by its commutators with the infinitesimal rotation operators. From (1.4) this reduces to the following commutator with an angular momentum

\[
\left[ J_{\ell}, T^k_{\ell} \right] = xT^k_{\ell},
\]

(31.4a)

\[
\left[ J_{\ell} \pm i J_{\ell} \right] = \left( \pm \frac{1}{2} \right) \left[ T^k_{\ell} \pm i T^k_{\ell} \right].
\]

(31.4b)

These commutators are naturally identical with the commutators of the spherical functions \( Y_{\ell m} \), since the transformation properties of these quantities are the same. This considerably simplifies the calculation of matrix elements of irreducible tensors as compared to those of reducible ones.

In the main, contraction, symmetrisation and anti-symmetrisation are the processes used to decompose a reducible tensor into a set of irreducible ones. The details of this reduction may be found in manuals on the theory of group representations (e.g., Gel’fand et al., 1958/9). We will give a summary of the results for the cases of greatest practical importance, namely \( \ell = 0, 1, 2 \). For \( \ell = 0 \) we have the scalar

\[
T = T^0_0
\]

(31.5)

For \( \ell = 1 \) (vector) the tensor representation \( \mathcal{U} = \mathcal{V} \) and is therefore equivalent to \( \mathcal{D}_0 \). We have

\[
T^1_1 = T^1_{-1} = \frac{1}{\sqrt{2}} (T^*_{\alpha \beta} + iT^*_{\beta \alpha})
\]

(31.6)

For \( \ell = 2 \) (second rank tensor) \( \mathcal{U} = \mathcal{O} \times \mathcal{V} \) and is therefore equivalent to \( \mathcal{D}_0 + \mathcal{D}_2 + \mathcal{D}_2 \). A reducible tensor of the second rank can thus be decomposed into irreducible tensors of ranks \( k = 0, 1, 2 \). Their components are given by

\[
T^2_0 = -\frac{1}{\sqrt{3}} (T_{\alpha \beta} + T_{\beta \alpha} + T_{\gamma \delta})
\]

(31.7a)

\[
T^2_{\pm 1} = \frac{i}{\sqrt{2}} (T_{\alpha \beta} - T_{\beta \alpha})
\]

(31.7b)

\[
T^2_{\pm 2} = \frac{1}{2} \left[ T^*_{\alpha \beta} - T_{\alpha \beta} \pm i (T^*_{\alpha \beta} + T_{\alpha \beta}) \right]
\]

(31.7c)

The normalisation in (31.6) and (31.7) is so chosen that

\[
\sum_{\ell, m} T^k_{\ell m} T^{k*}_{\ell m} = \sum_{m} (-1)^m T^k_{\ell m} T^{k*}_{\ell m} = \sum_{m} \langle m | m \rangle.
\]

(31.8)

Let us consider the matrix of the irreducible tensor \( T^k \) in the system of eigenfunctions of angular momentum

\[
(\phi | j m | T^k | a' j' m').
\]

Here \( a \) is the set of quantum numbers which completely specifies the set. The dependence of this matrix element on \( m, m' \) and \( q \) is fully determined by the transformation (geometrical) properties of the tensor \( T^k \), whence it follows (Wigner 1931, Racah 1942/12) that (the Wigner–Eckart theorem)

\[
(\phi | j m | T^k | a' j' m') = (-1)^{j m} \left( \sum_{q} \langle j j' | q m q m' \rangle (\phi | q m | T^k | a' j' m') \right).
\]

(31.9)

The above is the fundamental formula in the algebra of tensor operators. The phase has been so chosen that when the graphical methods of Chapter III are used one automatically obtains the phase factors required there. The quantity \( \langle \phi | T^k | \phi \rangle \) is independent of \( m, m' \) and \( q \) and is called the reduced matrix element of the tensor \( T^k \). The set of these quantities forms the reduced matrix of the tensor. For the reduced matrix of the hermitian conjugate tensor one readily finds (Racah 1942/12)

\[
(\phi | T^k | a' j' m') = (-1)^{j m} \langle a' j' | T^k | a m \rangle.
\]

(31.10)

If the tensor is hermitian (Racah 1942/12), then

\[
T^k_{a a} = (-1)^{a} T^k_{a a}
\]

(31.11)

and (31.10) reduces to

\[
(\phi | T^k | a' j') = (-1)^{j m} \langle a' j' | T^k | a j \rangle.
\]

(31.12)

i.e., the reduced matrix is not hermitian. It is useful to bear in mind that for the scalar operator \( T^0 \)

\[
(\phi | T^0 | a' j' m') = \delta (j, j') \delta (m, m') \langle j j' | a a' \rangle (\phi | a a' | T^0 | a' j' m').
\]

(31.9a)

If the scalar operator is \( T^0 = 1 \), then its diagonal element is equal to unity and from (31.9a) we find

\[
(\phi | T^0 | a a') = \delta (a, a') \langle j j' | a a' \rangle (\phi | a a' | T^0 | a a').
\]

(31.13)

A similar method is generally used for the calculation of reduced matrix elements. Using the definition of the operator, one calculates the simplest matrix element directly; the reduced matrix element can then be found from (31.9).

32. Tensor products

Let us consider the two irreducible tensors \( T^k \) and \( U^k \). Multiplying all the \( 2k_1 + 1 \) components of the first tensor by all the \( 2k_2 + 1 \) components of the second, we obtain the \( (2k_1 + 1) \times (2k_2 + 1) \) quantities \( (T^k \times U^k)_{\ell m} \). These quantities are components of the tensor \( (T^k \times U^k)_{\ell m} \) which is the direct product of the tensors \( T^k \) and \( U^k \). Its components transform according to \( \mathcal{D}_k \times \mathcal{D}_{-k} \) and this tensor.
therefore decomposes into irreducible tensors of ranks $k = h_1 - h_2, \cdots, h_1 + h_2$, which are called tensor products of rank $h$. In complete analogy with the coupling of angular momenta, the components of this product are given by the following formula:

$$
\left[ T^h \times U^k \right]_q = \sum_{m} T^h_m U^k_m (h_1 h_2 h_3 h_4 h_5) = (h_1 h_2 h_3 h_4 h_5).
$$

(32.1)

It is easily verified that this tensor satisfies the conditions (31.4).

Using the unitarity property of the matrix of the Clebsch-Gordan coefficients, we obtain

$$
T^h U^k = \sum_{m} \left[ T^h \times U^k \right]_m (h_1 h_2 h_3 h_4 h_5).
$$

(32.2)

The rank of the product can be zero only if $h_1 = h_2$. Then (32.1) becomes

$$
\left[ T^h \times U^k \right]_q = \sum_{m} (h_1 h_2 h_3 h_4 h_5) T^h m U^k m.
$$

(32.3)

Using (4.3b), we obtain further

$$
\left[ T^h \times U^k \right]_q = (-1)^q (h_1 h_2 h_3 h_4 h_5) \sum_{m} (-1)^m T^h m U^k m.
$$

(32.4)

This quantity is a scalar, and while it would have been natural to regard it as the definition of the scalar product of two tensors of the same rank, the traditional definition of a scalar product is not identical with (32.4), and is related to it in the following way:

$$
\left( T^h, U^k \right) = \sum_{m} (-1)^m T^h m U^k m = (-1)^q (h_1 h_2 h_3 h_4 h_5) \left( T^h \times U^k \right)_q.
$$

(32.5)

The advantage of the above definition over (32.4) is that for the case of vectors ($k = 1$) it coincides with the usual definition of the scalar product of two vectors; this can readily be seen by introducing (31.6) in (32.5).

For $h_1 = h_2 = 1$ and $k = 1$ the definition (32.1) is not identical with the usual vector product of two vectors. The following relation may easily be verified

$$
[A^i B^j] = -i \sqrt{2} \left[ A^i \times B^j \right].
$$

(32.6)

Here $A$ and $B$ are vectors and by $A^i$ and $B^j$ are meant the same vectors in the spherical basis (31.6). The left-hand side of (32.6) is the vector product of the given vectors.

Making use of the symmetry of the Clebsch-Gordan coefficients, one can readily verify the following relation

$$
\left[ T^h \times U^k \right]_{q} = (-1)^{h+k+q} \left[ T^q \times U^k \right]_{q},
$$

(32.7)

provided $T^h$ and $U^k$ commute and are hermitian, i.e., satisfy (31.11). The above equality shows that the tensor product of two hermitian tensors is hermitian when and only when $h_1 + h_2 + k$ is an even integer; otherwise it is an anti-hermitian tensor. One such tensor is, for example, the product $[A^i \times B^j]^\dagger$.

Generalizing (32.1) to the case of an arbitrary number of tensors, we write

$$
\left[ T^h_1 \times \cdots \times T^h_n \right]_{q} = \sum_{m} T^h_1 m U^h_2 m \cdots U^h_n m \left( h_1 h_2 h_3 h_4 h_5 \right) K^h_{q, m}.
$$

(32.8)

Here instead of the usual Clebsch-Gordan coefficient we have the corresponding generalised coefficient defined in (6.16). $K$ is the set of ranks of intermediate products, the number of which is $n - 2$. Generalization of (32.2) brings us to the following formula:

$$
\left[ T^h_1 \times \cdots \times T^h_n \right]_{q} = \sum_{K} \left[ T^h_1 \times \cdots \times T^h_n \right]_{q} K^h_{K, q, m} \left( h_1 h_2 h_3 h_4 h_5 \right) K^h_{q, m}.
$$

(32.9)

The concept of a scheme of addition of angular momenta introduced in section 6 must be replaced by the concept of a scheme of multiplication of tensors (or of addition of ranks of component tensors) in the case of multiplication of tensors. The passage from one scheme of multiplication to another is carried out exactly as for the eigenfunctions of coupledangular momenta in accordance with (8.1), i.e., with the aid of the transformation matrix. As we know, the elements of the latter may be expressed in terms of the $j$-coefficients.

The algebra described above is valid independently of what coordinates are acted upon by the individual tensor operators in the product. When calculating matrix elements it is convenient to express products of tensor operators in a form in which the operators acting on the same coordinates are adjacent to each other. This may be accomplished by changing the scheme of multiplication of the tensors with the aid of a transformation matrix.

**33. Expressions for matrix elements of products of tensor operators**

Let us consider the matrix element of the product (32.1), in which the two operators act on the same coordinates. We have

$$(a j m \left[ T^h_1 \times U^k_1 \right]_{q} a' j' m') = \sum_{j q'} (h_1 h_2 h_3 h_4 h_5) j q' (a j m \left[ T^h_1 \times U^k_1 \right]_{q} a' j' m') \times
$$

$$
(a' j' m') \left[ U^h_1 \times U^k_1 \right]_{q} a' j' m'.
$$

(33.1)

We apply (3.2) to the Clebsch-Gordan coefficient and (31.9) to the matrix elements. We sum over the three Wigner coefficients thus formed with the aid of (26.2) and obtain

$$(a j m \left[ T^h_1 \times U^k_1 \right]_{q} a' j' m') \times (b j k (j' m' q')) \left[ U^h_1 \times U^k_1 \right]_{q} a' j' m') \times$$

$$
\sum_{j q'} (a j m \left[ T^h_1 \times U^k_1 \right]_{q} a' j' m') \left( b j k \left( j' m' q' \right) \right) \times
$$

$$
(a' j' m') \left[ U^h_1 \times U^k_1 \right]_{q} a' j' m'.
$$

(33.2)
Comparing (33,2) with (31,9), we obtain the formula for the reduced matrix element of the product
\[
(a_j^i || T^h \times U^h || a_j^i) = (a_j^i || (-1)^j \cdot J' H^{-i} \times \\
\times \sum_{a_j} (a_j^i || T^h || a_j^i) (a_j^i || U^h || a_j^i) [h_j, h_j, h_j]_{j' j' j'}. \tag{33,3}
\]
In particular, for the reduced matrix element of the scalar product we have
\[
(a_j^i || T^h \times U^h || a_j^i) = 0 (J, j', (j')^2 \frac{1}{j} \times \\
\times \sum_{a_j} (-1)^{j' - j} (a_j^i || T^h || a_j^i) (a_j^i || U^h || a_j^i). \tag{33,4}
\]
Let us now turn to the tensor product (32,1) in which \( T^h \) acts on the coordinates 1 and \( U^h \) on the coordinates 2. In such cases the eigenfunctions \( a_{j^i}^m \) are usually constructed by coupling the angular momenta \( j_1 \) and \( j_2 \), the eigenfunctions of which depend on the corresponding coordinates. In this case
\[
(a_j^i a_{j^i}^m || T^h \times U^h || a_j^i a_{j^i}^m) = \\
= \sum_{a_{j^i}^m a_j^m} \langle j_1 j_2 j_1 ; j_2 j_2 m_2 || a_{j^i}^m a_j^m \rangle \times \\
\times \{ h_j, h_j, h_j || h_j, h_j, h_j \}_{j_1 j_2 j_1 \times j_2 j_2 m_2} \times \\
\times (a_j^i a_{j^i}^m || T^h || a_j^i a_{j^i}^m) (a_j^i a_{j^i}^m || U^h || a_j^i a_{j^i}^m). \tag{33,5}
\]
Proceeding as in (33,1), we obtain a sum of products of five Wigner coefficients. Summing it as in sections 26 and 27, we obtain
\[
(a_j^i a_{j^i}^m || T^h \times U^h || a_j^i a_{j^i}^m) = (\langle 0 || J || 0 \rangle)^2 \times \\
\times (a_j^i a_{j^i}^m || T^h || a_j^i a_{j^i}^m) (a_j^i a_{j^i}^m || U^h || a_j^i a_{j^i}^m) \times \\
\times (-1)^{j' - j} \langle j_1 j_2 h_j || \rangle_{j_1 j_2 h_j}. \tag{33,6}
\]
Comparing (33,6) and (31,9) we obtain the formula for the reduced matrix element of a tensor product the factors of which act on different coordinates
\[
(a_j^i a_{j^i}^m || T^h \times U^h || a_j^i a_{j^i}^m) = (\langle 0 || J || 0 \rangle)^2 \times \\
\times (a_j^i a_{j^i}^m || T^h || a_j^i a_{j^i}^m) (a_j^i a_{j^i}^m || U^h || a_j^i a_{j^i}^m). \tag{33,7}
\]
For the scalar product \( (k = 0) \) the above formula can be simplified:
\[
(a_j^i a_{j^i}^m || T^h \times U^h || a_j^i a_{j^i}^m) = (\langle 0 || J || 0 \rangle)^2 \times \\
\times (j_1 j_2 j_1 || j_2 j_2 j_2 \times a_{j^i}^m a_{j^i}^m) \times \\
\times (a_j^i a_{j^i}^m || T^h || a_j^i a_{j^i}^m) (a_j^i a_{j^i}^m || U^h || a_j^i a_{j^i}^m). \tag{33,8}
\]
From (33,7) one can obtain convenient formulas for the reduced matrix element of the operator which acts only on the coordinate 1. To do this one must set \( U^h = 1 \), \( h_j = 0 \), \( h_j = k \) and make use of (31,13). The result will be
\[
(a_j^i a_{j^i}^m || T^h || a_j^i a_{j^i}^m) = 0 (a_j^i a_{j^i}^m || (j_1 j_2 j_2 || j_2 j_2 j_2) \times \\
\times (a_j^i a_{j^i}^m || T^h || a_j^i) (a_j^i a_{j^i}^m || U^h || a_j^i). \tag{33,9}
\]
Analogously, for the operator which acts only on the coordinate 2, we will find
\[
(a_j^i a_{j^i}^m || U^h || a_j^i a_{j^i}^m) = 0 (a_j^i a_{j^i}^m || (j_1 j_2 j_2 || j_2 j_2 j_2) \times \\
\times (a_j^i a_{j^i}^m || U^h || a_j^i) (a_j^i a_{j^i}^m || U^h || a_j^i). \tag{33,10}
\]
The last two formulas correspond to (44a, b) in Racah's work /1942/. In the first case the operator commutes with \( J_3 \) and in the second case with \( J_1 \).

The fundamental formulas of the algebra of tensor operators are (31,9), (33,3) and (33,7). All the rest are particular cases of these. When a large number of tensors are multiplied it is not expedient to look for general formulas for the matrix elements, as in such cases the calculation is carried out by repeated application of the formulas for the product of two tensors. Examples of such calculations are given in the next section.

34. Calculation of matrix elements of complex products of tensor operators

In this section we shall consider concrete examples of complex tensor products, that is to say, products of more than two tensors. We shall choose examples which are characteristic of the calculation of matrix elements. We will therefore not be interested in the radial parts of the operators and will not write them explicitly.

Let us take a product of the following form
\[
[a_j^i a_{j^i}^m \times a_j^i a_{j^i}^m \times a_j^i a_{j^i}^m] = \left[ \begin{array}{c}
\hat{a}_j^i \times \hat{a}_j^j \times \hat{a}_j^k
\end{array} \right]. \tag{34,1}
\]
Here the supplementary superscripts \( i, j, l, m \) indicate the coordinates which are acted upon by the corresponding operators. We assume at first that \( i = j = 1 \), and
Next using (33.7) and then (33.3) we obtain

\[
\begin{aligned}
(a_{i,j} a_{i',j'}) & \equiv \left[ \left[ \hat{T} h \times \hat{U} h \right] \hat{v} h \times \hat{w} h \right] \left[ \hat{a}_{i,j} \hat{a}_{i',j'} \right] \\
& = \left( \delta_{i,j} \delta_{i',j'} \right) \left[ \delta_{j',i'} \left( h_{a2} h_{a3} \right) \right] \left[ (1 \pm i \pm j + h_{+} \pm i' \pm j' + h_{-}) \right] \\
& \times \sum_{a_{i,j}} \left( a_{i,j} \left\| T h \right\| a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \left\| a_{i',j'} \right\| a_{i',j'} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left\{ \begin{array}{c}
h_{a2} h_{a3} \\
j \ j' \ k_{a2} \ k_{a3}
\end{array} \right\}
\end{aligned}
\]

(33.2)

If \( i = l = 1 \) and \( j = m = 2 \), then it is convenient to change the order of the multiplication of the tensors so that the operators which act on the same coordinates are multiplied immediately. With the aid of (23.7) we obtain

\[
\begin{aligned}
& \left[ \left[ \hat{T} h \times \hat{U} h \right] \hat{v} h \times \hat{w} h \right] \\
& = \sum_{a_{i,j}} \left( h_{a2} h_{a3} \right) \left[ \delta_{j',i'} \left( h_{a2} h_{a3} \right) \right] \\
& \times \left[ \left[ \hat{T} h \times \hat{v} h \right] \hat{v} h \times \hat{w} h \right] \\
& \times \left[ \left[ \hat{T} h \times \hat{v} h \right] \hat{v} h \times \hat{w} h \right] \\
& \times \left\{ \begin{array}{c}
h_{a2} h_{a3} \\
j \ j' \ k_{a2} \ k_{a3}
\end{array} \right\}
\end{aligned}
\]

(33.3)

To find the expression for the reduced matrix element of (33.4), we first make use of (33.7) and then apply (33.3) to the individual elements obtained. We find

\[
\begin{aligned}
(a_{i,j} a_{i',j'}) & \equiv \left[ \left[ \hat{T} h \times \hat{U} h \right] \hat{v} h \times \hat{w} h \right] \left[ \hat{a}_{i,j} \hat{a}_{i',j'} \right] \\
& = \sum_{a_{i,j}} (-1)^{h_{+} + j' + h_{-} + j + k_{a2} + k_{a3}} \left( h_{a2} h_{a3} \right) \left[ \delta_{j',i'} \left( h_{a2} h_{a3} \right) \right] \\
& \times \left( a_{i,j} \left\| T h \right\| a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \left\| a_{i',j'} \right\| a_{i',j'} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left\{ \begin{array}{c}
h_{a2} h_{a3} \\
j \ j' \ k_{a2} \ k_{a3}
\end{array} \right\}
\end{aligned}
\]

(33.4)

Summation over \( h_{a2} \) and \( h_{a3} \) by the graphical methods leads to the following result:

\[
\begin{aligned}
(a_{i,j} a_{i',j'}) & \equiv \left[ \left[ \hat{T} h \times \hat{U} h \right] \hat{v} h \times \hat{w} h \right] \left[ \hat{a}_{i,j} \hat{a}_{i',j'} \right] \\
& = \sum \left( (-1)^{h_{+} + j + j' + h_{-} + k_{a2} + k_{a3}} \left( h_{a2} h_{a3} \right) \left[ \delta_{j',i'} \left( h_{a2} h_{a3} \right) \right] \right) \\
& \times \left( a_{i,j} \left\| T h \right\| a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left\{ \begin{array}{c}
h_{a2} h_{a3} \\
j \ j' \ k_{a2} \ k_{a3}
\end{array} \right\}
\end{aligned}
\]

(33.5)

In the numerical evaluation of this sum one should make use of the expansion (20.3) for the 15 \( f \)-coefficient of the third kind.

If in (33.4) \( i = j = l = 1 \) and \( m = 2 \), then the product should be transformed as follows:

\[
\begin{aligned}
& \left[ \left[ \hat{T} h \times \hat{U} h \right] \hat{v} h \times \hat{w} h \right] \\
& = \sum \left( (-1)^{h_{+} + j + j' + h_{-} + k_{a2} + k_{a3}} \left( h_{a2} h_{a3} \right) \left[ \delta_{j',i'} \left( h_{a2} h_{a3} \right) \right] \right) \\
& \times \left( a_{i,j} \left\| T h \right\| a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left\{ \begin{array}{c}
h_{a2} h_{a3} \\
j \ j' \ k_{a2} \ k_{a3}
\end{array} \right\}
\end{aligned}
\]

(33.6)

Here the transformation matrix is simplified in accordance with (9.2) and expressed in terms of the \( d f \)-coefficient. Applying (33.7) once and (33.3) twice we find

\[
\begin{aligned}
(a_{i,j} a_{i',j'}) & \equiv \left[ \left[ \hat{T} h \times \hat{U} h \right] \hat{v} h \times \hat{w} h \right] \left[ \hat{a}_{i,j} \hat{a}_{i',j'} \right] \\
& = \sum \left( (-1)^{h_{+} + j + j' + h_{-} + k_{a2} + k_{a3}} \left( h_{a2} h_{a3} \right) \left[ \delta_{j',i'} \left( h_{a2} h_{a3} \right) \right] \right) \\
& \times \left( a_{i,j} \left\| T h \right\| a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left( a_{i,j} \left\| V h \right\| a_{i,j} \left\| W h \right\| a_{i,j} \right) \\
& \times \left\{ \begin{array}{c}
h_{a2} h_{a3} \\
j \ j' \ k_{a2} \ k_{a3}
\end{array} \right\}
\end{aligned}
\]

(33.7)
Summing over $h_{32}$, we obtain

\[
(a_{j1} a_{j2} T_{32}^{ab}) \left[ \begin{array}{c} n_i' \times n_i'' \times n_i'' \times n_i'' \times n_i'' \times n_i'' \\
(\Psi^a_i \times \Psi^b_i) \times \left( \begin{array}{c} n_i' \times n_i'' \times n_i'' \times n_i'' \times n_i'' \times n_i'' 
\end{array} \right)
\right] = \left( -1 \right)^{n_i' + n_i'' + n_i' + n_i'' + n_i'' + n_i''} \times
\]

\[
\times \left( \begin{array}{c} h_1 h_2 h_3 \\
(\Psi^a_i \times \Psi^b_i) \times \left( \begin{array}{c} n_i' \times n_i'' \times n_i'' \times n_i'' \times n_i'' \times n_i'' 
\end{array} \right)
\right) \times
\]

\[
\times \left( \begin{array}{c} \Psi^a_i \times \Psi^b_i \times \Psi^a_i \times \Psi^b_i \times \Psi^a_i \times \Psi^b_i \\
(\Psi^a_i \times \Psi^b_i) \times \left( \begin{array}{c} n_i' \times n_i'' \times n_i'' \times n_i'' \times n_i'' \times n_i'' 
\end{array} \right)
\right) \times
\]

\[
\times (a_{j1} a_{j2} T_{32}^{ab} \Psi^a_i \times \Psi^b_i) = \left( -1 \right)^{n_i' + n_i'' + n_i' + n_i'' + n_i'' + n_i''} \times
\]

\[
\times (a_{j1} a_{j2} T_{32}^{ab} \Psi^a_i \times \Psi^b_i) = \left( -1 \right)^{n_i' + n_i'' + n_i' + n_i'' + n_i'' + n_i''} \times
\]

\[
\times (a_{j1} a_{j2} T_{32}^{ab} \Psi^a_i \times \Psi^b_i).
\]

In the phase factors of the above formulas we took into account the fact that the ranks of the tensors are integers.

35. Double tensors, their products and matrix elements

In many cases it is convenient to introduce the Kronecker product $T_{32}^{ab}$ with components $T_{32}^{ab}$, where $a_i = h_1, \ldots, h_2; b_i = h_1, \ldots, h_2$. The $2h_1 + 1$ components of this tensor with fixed $a_i$ and different $b_i$ form the basis of the Kronecker representation $\mathcal{D}_{h_1}$ upon rotation of one space, the $2h_2 + 1$ components with fixed $b_i$ and different $a_i$ transform according to the Kronecker representation $\mathcal{D}_{h_2}$ upon rotation of another space. For the sake of brevity we write $T_{32}^{ab}$ as a reduced tensor of rank $h_1$ with respect to $J_1$ and an irreducible tensor of rank $h_2$ with respect to $J_2$. $h_1$ and $h_2$ are usually integers. Instead of (31.9) we have for this tensor

\[
(a_{j1} a_{j2} T_{32}^{ab} \Psi^a_i \times \Psi^b_i) = \left( -1 \right)^{n_i' + n_i'' + n_i' + n_i'' + n_i'' + n_i''} \times
\]

\[
\times (a_{j1} a_{j2} T_{32}^{ab} \Psi^a_i \times \Psi^b_i).
\]

Upon simultaneous rotation of the two spaces, all the $(2h_1 + 1)(2h_2 + 1)$ components $T_{32}^{ab}$ transform according to the reducible representation $\mathcal{D}_{h_1} \times \mathcal{D}_{h_2}$. Therefore a tensor which is irreducible with respect to $J_1$ and $J_2$ is reducible with respect to

\[
J = J_1 + J_2.
\]

If we reduce $\mathcal{D}_{h_1} \times \mathcal{D}_{h_2}$, i.e., pass to the quantities

\[
T_{32}^{ab} = \sum_{\mathcal{D}_{h_1}} T_{32}^{ab}(\mathcal{D}_{h_1})(a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i),
\]

the double tensor $T_{32}^{ab}$ becomes the set of double tensors $T_{32}^{ab}$ of ranks $h = |h_1 - h_2|, \ldots, |h_1 + h_2|$, reduced with respect to $J$. Let us obtain the formula which relates the reduced matrix element of the reduced double tensor to the reduced matrix element of the non-reduced double tensor which appears in (35.1). As (35.1) is analogous to (32.1), proceeding literally in the same way as in the derivation of (33.7) we find

\[
(a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i) = \left( J J' J'' \right) \left( J J' J'' \right) \times
\]

\[
\times (a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i) = \left( J J' J'' \right) \left( J J' J'' \right) \times
\]

\[
\times (a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i).
\]

The similarity of (35.3) and (35.7) is due to the fact that we may, if we wish, regard $T_{32}^{ab} \Psi^a_i \times \Psi^b_i$ as a reduced double tensor of rank $h_1$ with respect to $J_1$ and $h_2$ with respect to $J_2$. The non-reduced double tensor in this case is the indefinite (dyadic) product $T_{32}^{ab} \Psi^a_i \times \Psi^b_i$. The internal structure of double tensors can be more general than that of tensor products. Thus, it may be a linear combination of such tensor products.

The tensor product of two irreducible double tensors is defined in the same way as in the case of the usual tensors. By analogy with (32.1) we write

\[
T_{32}^{ab} \Psi^a_i \times \Psi^b_i = \sum_{\mathcal{D}_{h_1} \times \mathcal{D}_{h_2}} T_{32}^{ab} \Psi^a_i \times \Psi^b_i \left( J J' J'' \right) \left( J J' J'' \right) \times
\]

\[
\times (a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i).
\]

To find the reduced matrix element of the product (35.4) we make use of (33.7) and apply (35.3) to the individual reduced matrix elements obtained. We then find

\[
(a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i) = \left( J J' J'' \right) \left( J J' J'' \right) \times
\]

\[
\times (a_{j1} a_{j2} \Psi^a_i \times \Psi^b_i).
\]

In accordance with the expression (20.3), for the $15j$-coefficient the above formula
can be written as follows
\[
\langle a | i | f | j | i' | f' \rangle = \left( \delta_{ij} \delta_{ij'} \right) \sum_{\sigma_f \sigma_{i'}} \left( \begin{array}{c} \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ \sigma_f \sigma_{i'} \end{array} \right) \prod_{\mu} \left( \begin{array}{c} h \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ h' \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ h \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ h' \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\
\end{array} \right) \times \left( \begin{array}{c} \lambda \mu \lambda \mu' \lambda \mu' \lambda \mu' \\ \sigma_f \sigma_{i'} \end{array} \right) \prod_{\mu} \left( \begin{array}{c} h \lambda \lambda' \lambda \lambda' \lambda \lambda' \\ h' \lambda \lambda' \lambda \lambda' \lambda \lambda' \\ h \lambda \lambda' \lambda \lambda' \lambda \lambda' \\ h' \lambda \lambda' \lambda \lambda' \lambda \lambda' \\
\end{array} \right).
\]
(35.5)

This formula is the analogue of (33.7) for double tensors. If \( K = 0 \), then, taking (32.5), (16.5) and (28.1) into account, we obtain the following expression from (35.5):
\[
\langle a | i | f | j | i' | f' \rangle = \times \sum_{\alpha_f \alpha_{i'}} \left( \begin{array}{c} \kappa_k \kappa_k' \kappa' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ \alpha_f \alpha_{i'} \end{array} \right) \prod_{\mu} \left( \begin{array}{c} h \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ h' \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ h \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\ h' \kappa_k \kappa_k' \kappa \kappa' \kappa_k \kappa_k' \kappa \kappa' \\
\end{array} \right) \times \left( \begin{array}{c} \lambda \mu \lambda \mu' \lambda \mu' \lambda \mu' \\ \alpha_f \alpha_{i'} \end{array} \right) \prod_{\mu} \left( \begin{array}{c} h \lambda \lambda' \lambda \lambda' \lambda \lambda' \\ h' \lambda \lambda' \lambda \lambda' \lambda \lambda' \\ h \lambda \lambda' \lambda \lambda' \lambda \lambda' \\ h' \lambda \lambda' \lambda \lambda' \lambda \lambda' \\
\end{array} \right).
\]
(35.7)

This formula gives the expression for the reduced matrix element of the scalar product of two double tensors.

Appendix 1

Notations for the Wigner, 6j-, 9j- and allied coefficients

In this appendix we give the notations for the Clebsch-Gordan coefficients and the notations and forms of the Wigner, 6j- and 9j-coefficients used by various authors.

Notations for the Clebsch-Gordan coefficients:
- Condon and Shortley /1941/, Racah /1942/, et al;
- (\( j_1 j_2 m_1 | j_3 j_4 m_2 \)) = (\( j_1 j_2 | j_3 j_4 m_2 \));
- Wigner /1931/:
  \( 3j \_m \_m \_m \);
- Fok /1940/:
  \( cf_{m}^{j} (m_1, m_2); \)
- Van der Waerden /1938/, Landau and Lifshitz /1948/:
  \( G_{m}^{j} \_m \_m \);
- Boys and Salni /1954/:
  \( X(j_1 m_1, j_2 m_2, m_3); \)
- Alder /1952/:
  \( cf_{m}^{j} (m_1, m_2); \)
- Rose /1957/:
  \( G(j_1 j_2 j_3 m_1 m_2 m_3); \)
- Fano /1951/:
  \( \langle j_1 j_2 m_1 j_3 m_2 | j_4 m_4 \rangle ; \)
- Biedenharn /1952/, Redmond /1954/:
  \( G_{m}^{j} (j_1 j_2 m_1 m_2); \)

Notations and forms for the Wigner coefficients:

\( (j_1 j_2 j_3) = \) (A.1.1)

- Racah /1942/:
  \( \langle j_1 j_2 j_3 | m_1 m_2 m_3 \rangle = (a) \)
- Landau and Lifshitz /1948/:
  \( \langle j_1 j_2 j_3 | m_1 m_2 m_3 \rangle = \) (b)
- Fano /1951/:
  \( \langle j_1 j_2 m_1 j_3 m_2 | j_3 m_3 | 0 \rangle. \) (c)
Schwinger /1952/
= X(\frac{j_1}{l_1} \frac{j_2}{l_2} m_1 m_2 m_3).

Notations and forms for the 6 j -coefficients (Racah coefficients)
$$
\begin{pmatrix}
\frac{j_1}{l_1} \frac{j_2}{l_2} \\
\frac{j_3}{l_3}
\end{pmatrix}
=\begin{pmatrix}
\frac{j_1}{l_1} & \frac{j_2}{l_2} \\
\frac{j_3}{l_3}
\end{pmatrix}.
$$

(A,1.2)

Racah /1942/
= \(-1^{j_1+j_3+i_1+i_3} W(j_1, j_2, j_3, j_4).

Jahn /1951/
= \(-1^{j_1+j_3+i_1+i_3} \frac{j_3}{l_3} \frac{j_1}{l_1} U(j_1, j_2, j_3, j_4).

Boys and Sahni /1954/
= \left(\frac{j_3}{l_3} \frac{j_1}{l_1}\right)^{-\frac{1}{2}} U(j_1, j_2, j_3, j_4).

Banerjee and Saha /1954/
= \left(\begin{array}{c}
   j_1 \\
   j_2 \\
   j_3 \\
\end{array}\right) \frac{j_1}{l_1} \frac{j_2}{l_2} \frac{j_3}{l_3}.

(c)

(d)

The quantities U are the transformation matrices for three angular momenta
(see (23,1)).

Notations and forms for the 9 j -coefficient (Wigner's 9 j -coefficient)
$$
\begin{pmatrix}
\frac{j_1}{l_1} \frac{j_2}{l_2} \\
\frac{j_3}{l_3}
\end{pmatrix}
=\begin{pmatrix}
\frac{j_1}{l_1} & \frac{j_2}{l_2} \\
\frac{j_3}{l_3}
\end{pmatrix}.
$$

(A,1.3)

Fano /1951/
= \chi(\frac{j_1}{l_1} \frac{j_2}{l_2} \frac{j_3}{l_3}).

Schwinger /1952/
= \chi(\frac{j_1}{l_1} \frac{j_2}{l_2} \frac{j_3}{l_3}) S(j_1, j_2, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}).

Hope /Jahn and Hope 1954/
= \left(\frac{j_1}{l_1} \frac{j_3}{l_3}\right)^{-\frac{1}{2}} \chi(j_1, j_2, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}).

Kennedy and Cliff /1957/
= \left(\frac{j_1}{l_1} \frac{j_3}{l_3}\right)^{-\frac{1}{2}} A(j_1, j_2, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}).

Arima, Horie and Tanshe /1954/
= U(j_1, j_2, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}; j_3, \frac{j_3}{l_3}).

The quantities \(\chi\) and \(A\) are the transformation matrices for four angular momenta
(see (23,4) and (23,7)).

The notation for the 12 j -coefficient of the first kind is universal (formula (19.1)).
The various notations for the 12 j -coefficient of the second kind are given
in formulas (19.3)-(19.5).

Appendix 2

Algebraic formulas for the Clebsch-Gordan coefficients

In this section we give the algebraic formulas for the coefficient \(B_{j_3, m_3}(j_1, m_1, j_2, m_2)\)
which enters into the formulas for the Clebsch-Gordan coefficients as follows (see the end of section 4):

\[ (j_3, j_2, m_3 m_2 j_1, j_1, \frac{1}{2}) = A(j_3) \cdot B(j_3, m_3)(j_1, m_1). \]

where*

\[ A(j_3) = \frac{(2j_3 + 1)^{2j_3 + 1} (2j_3 + 1)^{2j_3 + 1}}{2j_3 + 1 + 2j_3 + 1}. \]

(L.2.1)

The coefficient B satisfies the following condition:

\[ B(j_3, m_3)(j_1, m_1) = (-1)^{j_3 - m_3} B(j_3, m_3)(j_1, m_1). \]

The formulas are therefore given only for non-negative values of \(m_3\):

<table>
<thead>
<tr>
<th>(m_3)</th>
<th>(j_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>(-\left(m_3 + \frac{1}{2}\right))</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>(\left(m_1 + m_3 + \frac{1}{2}\right))</td>
</tr>
</tbody>
</table>

* See formula (4.17)—Translator.
### $j_4 = 1$

<table>
<thead>
<tr>
<th>$m_2$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$-\left[2(j_4 + m)(j_4 - m)\right]^{\frac{1}{2}}$</td>
<td>$\left[(j_4 - m)(j_4 - m + 1)\right]^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>0</td>
<td>$2m$</td>
<td>$-\left[2(j_4 + m)(j_4 - m + 1)\right]^{\frac{1}{2}}$</td>
</tr>
<tr>
<td>1</td>
<td>$\left[2(j_4 + m + 1)(j_4 - m + 1)\right]^{\frac{1}{2}}$</td>
<td>$\left[(j_4 + m)(j_4 + m + 1)\right]^{\frac{1}{2}}$</td>
</tr>
</tbody>
</table>

### $j_4 = \frac{3}{2}$

<table>
<thead>
<tr>
<th>$m_2$</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{3}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{3}{2}$</td>
<td>$\left[2(j_4 + m - \frac{1}{2})(j_4 - m + \frac{1}{2})\right]^{\frac{3}{2}}$</td>
<td>$-\left[(j_4 - m + \frac{3}{2})^{\frac{3}{2}}\right]$</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>$-\left[(j_4 + m - \frac{1}{2})\right]^{\frac{3}{2}}$</td>
<td>$\left[(j_4 + m + \frac{3}{2})^{\frac{3}{2}}\right]$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-\left[(j_4 + m + \frac{3}{2})\right]^{\frac{3}{2}}$</td>
<td>$-\left[(j_4 + m - \frac{1}{2})\right]^{\frac{3}{2}}$</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>$\left[3(j_4 + m + \frac{3}{2})^{\frac{3}{2}}(j_4 - m + \frac{3}{2})\right]^{\frac{3}{2}}$</td>
<td>$\left[(j_4 + m + \frac{3}{2})^{\frac{3}{2}}\right]$</td>
</tr>
</tbody>
</table>

### $j_4 = 2$

<table>
<thead>
<tr>
<th>$m_2$</th>
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<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$-\left[(j_4 - m + 1)^{\frac{3}{2}}\right]$</td>
<td>$\left[(j_4 - m + 1)^{\frac{3}{2}}\right]$</td>
</tr>
<tr>
<td>1</td>
<td>$-\left[(j_4 - m)^{\frac{3}{2}}\right]$</td>
<td>$\left[(j_4 - m)^{\frac{3}{2}}\right]$</td>
</tr>
<tr>
<td>2</td>
<td>$\left[(j_4 - m)^{\frac{3}{2}}\right]$</td>
<td>$\left[(j_4 - m)^{\frac{3}{2}}\right]$</td>
</tr>
<tr>
<td>3</td>
<td>$-\left[(j_4 + m - 1)^{\frac{3}{2}}\right]$</td>
<td>$\left[(j_4 + m - 1)^{\frac{3}{2}}\right]$</td>
</tr>
<tr>
<td>4</td>
<td>$\left[(j_4 + m + 1)^{\frac{3}{2}}\right]$</td>
<td>$\left[(j_4 + m + 1)^{\frac{3}{2}}\right]$</td>
</tr>
</tbody>
</table>

### Note:

(Tables for $j_4 = 5/2$, 3, 7/2 and 4 are given separately at the end of the book.)
Appendix 3

Diagrams of $18f$-coefficients

In this appendix we give the (conventionally) standard forms of the two-dimensional diagrams of those $18f$-coefficients which do not decompose into simpler $f$-coefficients. There are 18 of these. They are denoted by capital letters corresponding to the algebraic formulas of the next appendix.

Figure A. 3.1

Figure A. 3.2

Figure A. 3.3

Figure A. 3.4

Figure A. 3.5

Figure A. 3.6

Figure A. 3.7

Figure A. 3.8
Appendix 4

Properties of 18 $f$-coefficients

In this appendix we give the symmetry properties of the 18 $f$-coefficients and their expressions in terms of simpler $f$-coefficients. The letters on the left-hand side of each equation indicate the corresponding diagrams of the 18 $f$-coefficients in appendix 3.

$$A \left\{ j_1, j_2, j_3, j_4, j_5, j_6 \right\} = \left\{ h_1, h_2, h_3, h_4, h_5, h_6 \right\}$$

$$= \left\{ f_1, f_2, f_3, f_4, f_5, f_6 \right\}$$

$$= \sum_x (s) (-1)^{n+1} \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} . \quad (A.4.1)$$

$$B \left[ j_1, j_2, j_3, j_4, j_5, j_6 \right] = \left[ h_1, h_2, h_3, h_4, h_5, h_6 \right]$$

$$= \left[ j_1, j_2, j_3, j_4, j_5, j_6 \right]$$

$$= \sum_x (x) (-1)^{n+1} \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} . \quad (A.4.2)$$

$$C \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} = \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\}$$

$$= \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\}$$

$$= \sum_x (x) (-1)^{n+1} \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} \times \left\{ j_1, h_2, h_3, h_4, h_5, h_6 \right\} . \quad (A.4.3)$$

$$\psi = \sum_{i=1}^{4} (j_i - j) + k_1 + k_2 - (k_1 + k_2 + 2x).$$
\[ D = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1' & j_2' & j_3' & j_4' \\ k_1' & k_2' & k_3' & k_4' \end{vmatrix} = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1' & j_2' & j_3' & j_4' \\ k_1' & k_2' & k_3' & k_4' \end{vmatrix} = \sum \binom{\alpha}{\beta} (-1)^{\beta} \begin{vmatrix} j_1 \times j_2 \times j_3 \times k_1 \\ j_1 \times j_2 \times j_3 \times k_1 \end{vmatrix} \times \begin{vmatrix} k_1' \times j_1' \times k_1' \times j_1' \\ k_1' \times j_1' \times k_1' \times j_1' \end{vmatrix}. \quad (A, 4.4) \]

\[ \psi = j_1 - j_4 + j_3 + j_2 + j_1' - j_4' + j_3' + j_2' - j_1 + k_1 + k_1' + x. \]

\[ E = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1' & j_2' & j_3' & j_4' \\ k_1' & k_2' & k_3' & k_4' \end{vmatrix} = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1' & j_2' & j_3' & j_4' \\ k_1' & k_2' & k_3' & k_4' \end{vmatrix} = (\alpha) (-1)^{\alpha} \begin{vmatrix} j_1 \times j_2 \times j_3 \times k_1 \\ j_1 \times j_2 \times j_3 \times k_1 \end{vmatrix} \times \begin{vmatrix} k_1' \times j_1' \times k_1' \times j_1' \\ k_1' \times j_1' \times k_1' \times j_1' \end{vmatrix}. \quad (A, 4.5) \]

\[ \varphi = \sum_{i=1}^{s} x_i + 2k + 2k'. \]

\[ \psi = j_1 - j_4 + j_3 + j_2 + j_1' - j_4' + j_3' + j_2' - j_1 + k_1 + k_1' + x. \]

\[ F = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{vmatrix} = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{vmatrix} = (\alpha) (-1)^{\alpha} \begin{vmatrix} j_1 \times j_2 \times j_3 \times k_1 \times k_2 \times k_3 \\ j_1 \times j_2 \times j_3 \times k_1 \times k_2 \times k_3 \end{vmatrix} \times \begin{vmatrix} k_1' \times j_1' \times k_1' \times j_1' \times k_1' \times j_1' \end{vmatrix}. \quad (A, 4.6) \]

\[ \varphi = \sum_{i=1}^{s} x_i + 2k + 2k'. \]

\[ \psi = j_1 - j_4 + j_3 - j_2 - j_1' + j_4' + j_3' - j_2' + j_1 + k_1 + k_1' + x. \]

\[ G = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1' & j_2' & j_3' & j_4' \\ k_1' & k_2' & k_3' & k_4' \end{vmatrix} = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1' & j_2' & j_3' & j_4' \\ k_1' & k_2' & k_3' & k_4' \end{vmatrix} = (\alpha) (-1)^{\alpha} \begin{vmatrix} j_1 \times j_2 \times j_3 \times k_1 \times k_2 \times k_3 \\ j_1 \times j_2 \times j_3 \times k_1 \times k_2 \times k_3 \end{vmatrix} \times \begin{vmatrix} k_1' \times j_1' \times k_1' \times j_1' \times k_1' \times j_1' \end{vmatrix}. \quad (A, 4.7) \]

\[ \varphi = \sum_{i=1}^{s} x_i + 2k + 2k'. \]

\[ \psi = j_1 - j_4 + j_3 - j_2 - j_1' + j_4' + j_3' - j_2' + j_1 + k_1 + k_1' + x. \]
\[ H = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \sum_{s} (x_s, x_s)(-1)^s \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 \end{bmatrix} \times \begin{bmatrix} x_s & x_s & x_s & x_s \end{bmatrix} \times \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 \end{bmatrix}. \] (A.4.8)

\[ \psi = \lambda_0 + \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + 2\lambda_6 - x_s + x_s \]

\[ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \sum_{s} (x_s, x_s)(-1)^s \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix}. \] (A.4.9)

\[ \psi = l_0 + l_1 + l_2 - l_3 + l_4 + l_5 + 2l_6 - x_s + x_s \]

\[ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \sum_{s} (x_s, x_s)(-1)^s \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix}. \] (A.4.10)

\[ \psi = h_0 - h_1 + h_2 - h_3 + h_4 + \sum_{s} (j_1, j_2) \]

\[ \psi = 2r_0 + 2j_0 \]

\[ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix} = \sum_{s} (x_s, x_s)(-1)^s \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \end{bmatrix}. \] (A.4.11)
\[ M = \begin{pmatrix} j_1 & l_1 & s & l_2 & j_2 \\ r & p_0 & p_0 & p_0 & m \\ j_1^* & l_1^* & s^* & l_2^* & j_2^* \end{pmatrix} = \begin{pmatrix} j_1 & l_1 & s & l_2 & j_2 \\ r & p_0 & p_0 & p_0 & m \\ j_1^* & l_1^* & s^* & l_2^* & j_2^* \end{pmatrix} \]

\[ \psi = j_1 - j_2 - j_1^* - j_2^* + k_1 + k_2 + k_1^* + k_2^* \]

\[ N = \begin{pmatrix} j_1 & h_1 & m & h_1^* & j_1^* \\ l_1 & p_0 & p_0^* & l_1^* \\ j_1^* & l_1^* & p_0 & p_0^* \end{pmatrix} = \begin{pmatrix} j_1 & h_1 & m & h_1^* & j_1^* \\ l_1 & p_0 & p_0^* & l_1^* \\ j_1^* & l_1^* & p_0 & p_0^* \end{pmatrix} \]

\[ \psi = j_1 - j_2 - j_1^* - j_2^* + k_1 + k_2 + k_1^* + k_2^* \]

\[ P = \begin{pmatrix} j & h_1 & l_1 & l_1^* & h_2 & j' \\ h_2 & m_1 & m_1^* & h_2 & j' \end{pmatrix} = \begin{pmatrix} j & h_1 & l_1 & l_1^* & h_2 & j' \\ h_2 & m_1 & m_1^* & h_2 & j' \end{pmatrix} \]

\[ \psi = j_1 + f + k_1 + k_1^* + m_1 + m_1^* + u + 2s \]

\[ \phi = j + f + k_1 + k_1^* + m_1 + m_1^* + u - x \]

\[ R = \begin{pmatrix} j & k & l & m & n & u \\ r & s & s' & r' & p & t' \end{pmatrix} = \begin{pmatrix} j & k & l & m & n & u \\ r & s & s' & r' & p & t' \end{pmatrix} \]

\[ \psi = m + f + t + r + x \]

\[ S = \begin{pmatrix} j & k & l & m & n & t \\ r & p & s & r' & t' \end{pmatrix} = \begin{pmatrix} j & k & l & m & n & t \\ r & p & s & r' & t' \end{pmatrix} \]

\[ \psi = j + f + k + l + m + r + s + t + x \]
Expressions for the transformation matrices of eigenfunctions of five coupled angular moments

For matrix elements which are expressed in terms of $6j$- and $9j$-coefficients, we only give their expressions in terms of the product of the simplified matrices. In the remaining cases we give expressions for the matrix elements in terms of $12j$-coefficients of the first and second kind. Expressions are given only for those matrices which were included in Table 23.1, as the remainder may be reduced to these by elementary means (cf. the remark in connection with Table 23.1 on page 79). The notation for products of matrices and coupling schemes was stipulated in section 21.

To shorten the notation the abbreviation $\{\ldots\}$ is used, in which the dotted line denotes the product of the quantities $(2j+1)$ for all intermediate angular moments on the right and left sides of the transformation matrix.

Matrices susceptible of expression in terms of the product of two $6j$-coefficients:

\[
\begin{align*}
(12345)^{h_{1}}(13254)^{h_{2}} & = \\
= & (12345)^{h_{1}}(12345)^{h_{2}}(13254)^{h_{2}}. & (A,5.1) \\
(12345)^{k_{1}}(35412)^{k_{2}} & = \\
= & (12345)^{k_{1}}(12354)^{k_{2}}(35412)^{k_{2}}. & (A,5.2)
\end{align*}
\]

Matrices susceptible of expression in terms of product of three $6j$-coefficients:

\[
\begin{align*}
(12345)^{h_{1}}(13452)^{k_{1}} & = \\
= & (12345)^{h_{1}}(12345)^{h_{2}}(13452)^{k_{1}}. & (A,5.3) \\
(12345)^{h_{1}}(13524)^{k_{1}} & = \\
= & (12345)^{h_{1}}(12345)^{h_{2}}(13524)^{k_{1}}. & (A,5.4) \\
(12345)^{h_{1}}(34152)^{k_{1}} & = \\
= & (12345)^{h_{1}}(12345)^{h_{2}}(34152)^{k_{1}}. & (A,5.5)
\end{align*}
\]
\[
\begin{align*}
(12345)^{4} & (13452)^{4} = \\
& = (12345)^{4}(12345)^{4}(12345)^{4}(13452)^{4}. \quad (A.5.6) \\
(12345)^{4} & (45312)^{4} = \\
& = (12345)^{4}(12345)^{4}(45312)^{4}. \quad (A.5.7) \\
(12345)^{4} & (35142)^{4} = \\
& = (12345)^{4}(12345)^{4}(35142)^{4}. \quad (A.5.8) \\
(12345)^{4} & (13245)^{4} = \\
& = (12345)^{4}(13245)^{4}. \quad (A.5.9) \\
& = (12345)^{4}(12345)^{4}(13245)^{4}(13245)^{4}. \quad (A.5.10) \\
(12345)^{4} & (13425)^{4} = \\
& = (12345)^{4}(13425)^{4}. \quad (A.5.11) \\
(12345)^{4} & (13452)^{4} = \\
& = (12345)^{4}(13452)^{4}. \quad (A.5.12) \\
(12345)^{4} & (15234)^{4} = \\
& = (12345)^{4}(15234)^{4}. \quad (A.5.13) \\
(12345)^{4} & (13452)^{4} = \\
& = (12345)^{4}(13452)^{4}. \quad (A.5.14) \\
(12345)^{4} & (13452)^{4} = \\
& = (12345)^{4}(13452)^{4}. \quad (A.5.15) \\
\end{align*}
\]

Matrices susceptible of expression in terms of products of 6j- and 9j-coefficients:

\[
\begin{align*}
(12345)^{4} & (13542)^{4} = \\
& = (12345)^{4}(12345)^{4}(13542)^{4}. \quad (A.5.16) \\
(12345)^{4} & (13452)^{4} = \\
& = (12345)^{4}(14352)^{4}. \quad (A.5.17) \\
(12345)^{4} & (13542)^{4} = \\
& = (12345)^{4}(15234)^{4}. \quad (A.5.18) \\
(12345)^{4} & (15234)^{4} = \\
& = (12345)^{4}(15234)^{4}. \quad (A.5.19) \\
\end{align*}
\]
\[
(12345)^{a_1}(13542)^{a_2} = (12345)^{a_1}(13542)^{a_2} \cdot (13542)^{a_3}.
\]
(A.5.26)

\[
(12345)^{a_1}(13245)^{a_2} = (12345)^{a_1}(13245)^{a_2} \cdot (13245)^{a_3}.
\]
(A.5.27)

Matrices susceptible of expression in terms of 12\(j\)-coefficients of the first kind:

\[
(12345)^{a_1}(14532)^{a_2} = (-1)^{y} \left[ \ldots \right]^{\frac{1}{2}} \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right) \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right)
\]
(A.5.28)

\[
\varphi = J_{s_{15}} - J_{s_{26}} + J_{s_{23}} - J_{s_{23}}.
\]

\[
(12345)^{a_1}(13542)^{a_2} = (-1)^{y} \left[ \ldots \right]^{\frac{1}{2}} \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right) \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right)
\]
(A.5.29)

\[
\varphi = J_{s_{15}} - J_{s_{26}} + J_{s_{23}} - J_{s_{23}}.
\]

\[
(12345)^{a_1}(15432)^{a_2} = (-1)^{y} \left[ \ldots \right]^{\frac{1}{2}} \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right) \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right)
\]
(A.5.30)

\[
\varphi = J_{s_{15}} - J_{s_{26}} + J_{s_{23}} - J_{s_{23}}.
\]

\[
(12345)^{a_1}(13542)^{a_2} = (-1)^{y} \left[ \ldots \right]^{\frac{1}{2}} \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right) \times \left( J_{s_{15}} \ J_{s_{26}} \ J_{s_{23}} \ J_{s_{23}} \right)
\]
(A.5.31)

\[
\varphi = J_{s_{15}} - J_{s_{26}} + J_{s_{23}} - J_{s_{23}}.
\]
\[
\begin{align*}
\psi &= j_1 - j_5 - J_6 + J_{1356} \\
(13245)_{(13254) \alpha} &= (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
\end{array} \right\} \\
& = \psi = j_1 - j_5 - J_6 + J_{1356} + J. \\
(12345)_{(13254) \alpha} &= (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
J_{156} & j_5 & j_1 \\
\end{array} \right\} \\
& = \psi = j_1 + j_5 - j_6 - J_{1356} + J_{156} + J_{156} - J.
\end{align*}
\]

Matrices susceptible of expression in terms of 12 \textit{j}-coefficients of the second kind:

\[
(12345)_{(14523) \alpha} = (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
\end{array} \right\} \\
& = \psi = j_1 + j_5 - j_6 - J_{1356} + J_{156} + J_{156} - J.
\]

\[
(12345)_{(13254) \alpha} = (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
\end{array} \right\} \\
& = \psi = j_1 - j_5 - J_{1356} - J_{156} + J.
\]

\[
(12345)_{(13254) \alpha} = (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
\end{array} \right\} \\
& = \psi = j_1 - j_5 - j_6 + J_{1356} + J_{156} - J.
\]

\[
(12345)_{(13425) \alpha} = (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
\end{array} \right\} \\
& = \psi = j_1 - j_5 - j_6 + J_{1356} + J_{156} - J.
\]

\[
(12345)_{(13524) \alpha} = (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
\end{array} \right\} \\
& = \psi = j_1 - j_5 - j_6 - J_{1356} + J_{156} + J_{156} - J.
\]

\[
(12345)_{(15324) \alpha} = (-1)^\alpha \{ \ldots \}^4 x \\
& \times \left\{ \begin{array}{c}
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
J_{156} & J_{156} & J_{156} \\
\end{array} \right\} \\
& = \psi = J_1 - j_6 - J_{156}.
\]

Appendix 6

Sum rules on \textit{j}-coefficients

In this appendix we give the most frequently used sum rules. We have for convenience repeated the sum rules for 6 \textit{j}-, 9 \textit{j}-, and 12 \textit{j}-coefficients which appeared in the text. For the sum rules on 3 \textit{nj}-coefficients, which were given in the text, we have simply indicated the numbers of these formulas. Wherever possible the formulas have been placed in such a way that the succession of formulas of the same type for 6 \textit{j}-, 9 \textit{j}- and 12 \textit{j}-coefficients ends with the general formula. In most cases we confined ourselves to formulas containing only one summation parameter.

Orthogonality relations:

\[
\sum_{\alpha} \left( \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{array} \right) = 0.
\]

\[
\sum_{\alpha} \left( \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{array} \right) = 0.
\]

Orthogonality relations:

\[
\sum_{\alpha} \left( \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{array} \right) = 0.
\]

Orthogonality relations:

\[
\sum_{\alpha} \left( \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{array} \right) = 0.
\]

Orthogonality relations:

\[
\sum_{\alpha} \left( \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
\end{array} \right) = 0.
\]
\[
\sum_{a_1 \ldots a_{n-1}} \left( \prod_{i=1}^{n-1} \left( \begin{array}{cccc}
 j_1 & x_1 & x_2 & \cdots & x_{n-1} \\
 j_2 & l_1 & l_2 & \cdots & l_{n-1} & l_n \\
 \vdots & \vdots & \vdots & & \vdots & \vdots \\
 j_n & k_1 & k_2 & \cdots & k_{n-1} & k_n \\
 \end{array} \right) \right) \times \\
\times \left( \begin{array}{cccc}
 j_1 & x_1 & x_2 & \cdots & x_{n-1} \\
 j_2 & l_1 & l_2 & \cdots & l_{n-1} & l_n \\
 \vdots & \vdots & \vdots & & \vdots & \vdots \\
 j_n & k_1 & k_2 & \cdots & k_{n-1} & k_n \\
 \end{array} \right) = \\
= 8 (k_2, k_3) \ldots \delta(k_n, x) \delta(l_{n-1}, \xi) (h_{n-1}^{-1}) \ldots (h_2^{-1}) (l_1^{-1})^{-1} \\
\times \{ h_{n-1} l_{n-1} \} \ldots \{ h_2 l_2 \} \{ h_1 l_1 \} \{ j_n l_n j_n \}. \tag{A. 6.5}
\]

See also (24.1).

Sum rules of the type in which the sum of the product of two \( j \)-coefficients reduces to the same \( j \)-coefficient:

\[
\sum_{x} (x)(-1)^{j_1 j_2 j_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] = (-1)^{j_1 l_1 l_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.6}
\]

\[
\sum_{n} (x_1)(x_2)(-1)^{j_1 j_2 j_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] = \\
= (-1)^{j_1 l_1 l_3 j_2 j_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.7}
\]

\[
\sum_{n,n'} (x_1)(x_2)(-1)^{j_1 j_2 j_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] \\
\times \left[ \begin{array}{ccc}
 j_1' & l_1' & l_3' \\
 j_2' & l_2' & l_3' \\
 j_3' & l_3' & l_3' \\
 \end{array} \right] = (-1)^{-l_1 l_2 l_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.8}
\]

\[
\sum_{n,n',n''} (x_1)(x_2)(-x_3) \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] = \\
= (k_3) \left[ \begin{array}{ccc}
 j_1 & j_2 & j_3 \\
 j_2 & j_3 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.9}
\]

See also (24.6).

Formulas for the expansion of \( j \)-coefficients:

\[
\left\{ \begin{array}{ccc}
 j_1 & j_2 & j_3 \\
 l_1 & l_2 & l_3 \\
 h_1 & h_2 & h_3 \\
 \end{array} \right\} = \sum_{x} (x)(-1)^{j_1 j_2 j_3} \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] \times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] \times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.10}
\]

\[
\left\{ \begin{array}{ccc}
 j_1 & j_2 & j_3 \\
 l_1 & l_2 & l_3 \\
 h_1 & h_2 & h_3 \\
 \end{array} \right\} = \\
= (-1)^{j_1 l_1 l_3 j_2 j_3} \sum_{x} (x) \left[ \begin{array}{ccc}
 h_1 & k_2 & x \\
 h_2 & k_3 & x \\
 h_3 & k_2 & x \\
 \end{array} \right] \times \\
\times \left[ \begin{array}{ccc}
 j_1 & k_2 & x \\
 j_2 & k_3 & x \\
 j_3 & k_2 & x \\
 \end{array} \right] \times \\
\times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.11}
\]

See also (17.1) and (17.2).

Recursion formulas:

\[
\left\{ \begin{array}{ccc}
 j_1 & j_2 & j_3 \\
 l_1 & l_2 & l_3 \\
 h_1 & h_2 & h_3 \\
 \end{array} \right\} = (-1)^{j_1 l_1 l_3} \times \\
\times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] \times \\
\times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right] \times \\
\times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 j_3 & l_3 & l_3 \\
 \end{array} \right]. \tag{A. 6.12}
\]

\[
\left\{ \begin{array}{ccc}
 j_1 & j_2 & \cdots & j_{n-1} & j_n \\
 l_1 & l_2 & \cdots & l_{n-1} & l_n \\
 h_1 & h_2 & \cdots & h_{n-1} & h_n \\
 \end{array} \right\} = (-1)^{j_1 l_1 l_3 h_2} \times \\
\times \sum_{x} (x) \left[ \begin{array}{ccc}
 h_2 & j_3 & x \\
 h_3 & j_4 & x \\
 \vdots & \vdots & \vdots \\
 h_n & j_2 & x \\
 \end{array} \right] \times \\
\times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 \vdots & \vdots & \vdots \\
 j_{n-1} & l_{n-1} & l_{n-1} \\
 \end{array} \right]. \tag{A. 6.13}
\]

\[
\left\{ \begin{array}{ccc}
 j_1 & j_2 & \cdots & j_{n-1} & j_n \\
 l_1 & l_2 & \cdots & l_{n-1} & l_n \\
 h_1 & h_2 & \cdots & h_{n-1} & h_n \\
 \end{array} \right\} = \\
= (-1)^{j_1 l_1 l_3 h_2} \times \\
\times \sum_{x} (x) \left[ \begin{array}{ccc}
 h_2 & j_3 & x \\
 h_3 & j_4 & x \\
 \vdots & \vdots & \vdots \\
 h_n & j_2 & x \\
 \end{array} \right] \times \\
\times \left[ \begin{array}{ccc}
 j_1 & l_1 & l_3 \\
 j_2 & l_2 & l_3 \\
 \vdots & \vdots & \vdots \\
 j_{n-1} & l_{n-1} & l_{n-1} \\
 \end{array} \right]. \tag{A. 6.14}
\]

See also (24.8).
Single sum of one $3n^j$-coefficient:
\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x).
\]  
(A. 6.15)

\[
\sum_x (-1)^3 (\xi_j \xi_x) = (-1)^{3j} (\xi_j).
\]  
(A. 6.16)

\[
\sum_x (-1)^{j+3} (\xi_j \xi_x) = (-1)^{j^2} (\xi_j).
\]  
(A. 6.17)

\[
\sum_x (-1)^{j^2} (\xi_j \xi_x) = (-1)^{-j-j^2} (\xi_j) (\xi_x)^{1/2} \times \delta(\xi_j, \xi_x).
\]  
(A. 6.18)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} (-1)^{j^2} \delta(\xi_j, \xi_x) \times \delta(\xi_j, \xi_x) \{ h_2 h_3 \}.
\]  
(A. 6.19)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.20)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.21)

\[
\sum_x (-1)^{j^2} (\xi_j \xi_x) = (-1)^{j^2 + h + h^2} \delta(\xi_j, \xi_x) \times \delta(\xi_j, \xi_x) \{ h_2 h_3 \}.
\]  
(A. 6.22)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.23)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.24)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.25)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.26)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.27)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.28)

\[
\sum_x (\xi_j \xi_x) = (\xi_j) (\xi_x)^{-1} \delta(\xi_j, \xi_x) \{ h_2 h_3 \} \{ h_2 h_3 \} \{ h_2 h_3 \}.
\]  
(A. 6.29)

See also (24,11).
\[
\sum_{x} \left\{ j_1, j_2, \ldots, j_{n-1}, x \right\} \\
= (-1)^{l_n - l_1 + \sum_{k=n-2}^{1} x_k} \left[ j_1, \ldots, j_{n-2}, j_{n-1} \vphantom{\sum} \right] \left[ l_1, \ldots, l_{n-2}, l_{n-1} \right] \vphantom{\sum} \left[ A_n \right] \vphantom{\sum} \left[ A_{n-1} \right] \left[ A_{n-2} \vphantom{\sum} \right] (A. 6.30) \\
\sum_{x} \left\{ j_1, j_2, \ldots, j_{n-1}, x \right\} \\
= (-1)^{l_n - l_1 + \sum_{k=n-2}^{1} x_k} \left[ j_1, \ldots, j_{n-2}, j_{n-1} \right] \left[ h_1, \ldots, h_{n-2}, h_{n-1} \right] \left[ A_n \right] (A. 6.31) \\
\sum_{x} (-1)^{l} \left\{ j_1, l_1, l_2, \ldots, l_{n-1}, l_{n-1} \right\} \\
= 8 \left( h_{n-2}, h_1 \right) \left( j_1 \right)^{-1} (-1)^{l_n - l_1 + \sum_{k=n-2}^{1} l_k} \left\{ j_1, h_1, l_{n-1} \right\} \left\{ x \right\} \left[ j_1, l_1, l_2, \ldots, l_{n-1} \right] \left[ h_1, \ldots, h_{n-2}, h_{n-1} \right] \left[ A_n \right] (A. 6.32) \\
\sum_{x} (-1)^{l} \left\{ j_1, j_2, \ldots, j_{n-2}, j_{n-1}, l_{n-1}, j_{n-1} \right\} \\
= 8 \left( h_{n-2}, h_1 \right) \left( k_1 \right)^{-1} (-1)^{l_n - l_1 + \sum_{k=n-2}^{1} j_k} \left\{ j_1, h_1, l_{n-1} \right\} \left\{ x \right\} \left[ j_1, j_2, \ldots, j_{n-2} \right] \left[ h_1, \ldots, h_{n-2}, h_{n-1} \right] \left[ A_{n} \right] (A. 6.33) \\
A single sum of products of one 3nj-coefficient and one 6j-coefficient:
\[
\sum_{x} \left\{ j_1, j_2, j_3 \right\} \left\{ x \right\} \left\{ h_1, h_2, h_3 \right\} \\
= (-1)^{l} \left\{ j_1, j_2, j_3 \right\} \left\{ l_1, l_2, l_3 \right\} \left\{ h_1, h_2, h_3 \right\} \left\{ j_1, j_2, j_3 \right\} \left\{ x \right\} \left\{ h_1, h_2, h_3 \right\} (A. 6.34) 
\]
\[ \sum_x (\alpha)(-1)^x \begin{bmatrix} j_1 & j_2 & j_3 & j_4 & \cdots & j_n & l_1 & l_2 & l_3 & \cdots & l_n \end{bmatrix} \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \cdots & h_n \end{bmatrix} = \]

\[ = (-1)^{y + 1 + l_1 + l_2 + l_3 + l_4 + \cdots + l_n} \times \left( \begin{bmatrix} j_1 & j_2 & j_3 & j_4 & \cdots & j_n & l_1 & l_2 & l_3 & \cdots & l_n \end{bmatrix} \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \cdots & h_n \end{bmatrix} \right). \]  

(A. 6.40)

\[ \sum_x (\alpha)(-1)^x \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} = \]

\[ = \left( \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} \right). \]  

(A. 6.41)

\[ \sum_x (\alpha) \left( \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} \right) = \]

\[ = (-1)^{n-1} \left( \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} \right). \]  

(A. 6.42)

\[ \sum_x (\alpha)(-1)^x \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} = \]

\[ = (-1)^{n-1 + (n-1)(y + 1 + l_1 + l_2 + l_3 + l_4 + \cdots + l_n)} \times \left( \begin{bmatrix} j_1 & j_2 & j_3 & j_4 & \cdots & j_n & l_1 & l_2 & l_3 & \cdots & l_n \end{bmatrix} \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \cdots & h_n \end{bmatrix} \right). \]  

(A. 6.43)

\[ \sum_x (\alpha)(-1)^x \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} = \]

\[ = (-1)^{n-1 + (n-1)(y + 1 + l_1 + l_2 + l_3 + l_4 + \cdots + l_n)} \times \left( \begin{bmatrix} j_1 & j_2 & j_3 & j_4 & \cdots & j_n & l_1 & l_2 & l_3 & \cdots & l_n \end{bmatrix} \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \cdots & h_n \end{bmatrix} \right). \]  

(A. 6.44)

\[ R_{n-1} = \sum_{i=1}^{n-1} (j_i + l_i + h_i) \]

\[ \sum_x (\alpha) \begin{bmatrix} j_1 & \cdots & j_{n-1} & l_{n-1} & l_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} = \]

\[ = (-1)^{y + h_{n-1} + h_n} \left( \begin{bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-2} & h_n \end{bmatrix} \right) \times \left( \begin{bmatrix} l_{n-1} & l_{n-1} & l_n \end{bmatrix} \begin{bmatrix} h_{n-1} & h_{n-1} & h_n \end{bmatrix} \right). \]  

(A. 6.45)

\[ \sum_x (\alpha) \begin{bmatrix} j_1 & \cdots & j_{n-1} & l_{n-1} & l_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} = \]

\[ = (-1)^{y + h_{n-1} + h_n} \left( \begin{bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-2} & h_n \end{bmatrix} \right) \times \left( \begin{bmatrix} l_{n-1} & l_{n-1} & l_n \end{bmatrix} \begin{bmatrix} h_{n-1} & h_{n-1} & h_n \end{bmatrix} \right). \]  

(A. 6.46)

A single sum of products of one 3j- and 9j- and more complicated coefficients:

\[ \sum_x (\alpha) \begin{bmatrix} j_1 & j_2 & l_1 & l_2 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & l_1 & l_2 \end{bmatrix} = (-1)^{y + h_{n-1} + h_n} \left( \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \end{bmatrix} \right). \]  

(A. 6.47)

\[ \sum_x (\alpha)(-1)^y \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} = \]

\[ \times \left( \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \end{bmatrix} \begin{bmatrix} h_1 & \cdots & h_{n-1} & h_n \end{bmatrix} \right) \times \left( \begin{bmatrix} l_{n+1} & l_{n+1} \end{bmatrix} \begin{bmatrix} h_{n+1} & h_{n+1} \end{bmatrix} \right). \]  

(A. 6.48)
\[
\sum_{s}(s)\left(-1\right)^s \left\{ f_1 \ldots f_{n-1} f_n \right\} \times \left[ \begin{array}{cccc}
 f_1 & f_n & \cdots & f_{n-1} \\
 h_1 & h_n & \cdots & h_{n-1} \\
 & & \ddots & \vdots \\
 & & & h_1
\end{array} \right] \times \left[ \begin{array}{cccc}
 l_1 & l_n & \cdots & l_{n-1} \\
 h_1 & h_n & \cdots & h_{n-1} \\
 & & \ddots & \vdots \\
 & & & h_1
\end{array} \right] 
\]

(6.49)

\[
\sum_{s}(s) \left\{ f_1 \ldots f_{n-1} f_n \right\} \times \left[ \begin{array}{cccc}
 f_1 \cdots f_{n-1} f_n & f_{n+1} \\
 h_1 \cdots h_{n-1} h_n & h_{n+1} \\
 & & \ddots & \vdots \\
 & & & h_1
\end{array} \right] 
\]

(6.50)

\[
\sum_{s}(s) \left\{ f_1 \ldots f_{n-1} f_n \right\} \times \left[ \begin{array}{cccc}
 f_1 \cdots f_{n-1} f_n & f_{n+1} \\
 h_1 \cdots h_{n-1} h_n & h_{n+1} \\
 & & \ddots & \vdots \\
 & & & h_1
\end{array} \right] 
\]

(6.51)

\[
\sum_{s}(s) \left\{ f_1 \ldots f_{n-1} f_n \right\} \times \left[ \begin{array}{cccc}
 f_1 \cdots f_{n-1} f_n & f_{n+1} \\
 h_1 \cdots h_{n-1} h_n & h_{n+1} \\
 & & \ddots & \vdots \\
 & & & h_1
\end{array} \right] 
\]

(6.52)

A few other sum rules:

\[
\sum_{s}(s)\left(-1\right)^s \left\{ f_1 \times f_1 \right\} \left\{ f_2 \times f_2 \right\} \left\{ f_3 \times f_3 \right\} = \left(-1\right)^{f_1+f_2+f_3+f_4+f_5+f_6+f_7+f_8} \left\{ h_1 \times h_2 \right\} \left\{ h_3 \times h_4 \right\} \left\{ h_5 \times h_6 \right\} \left\{ h_7 \times h_8 \right\} 
\]

(6.53)

\[
\sum_{s}(s)\left(-1\right)^s \left\{ f_1 \times f_1 \right\} \left\{ f_2 \times f_2 \right\} \left\{ f_3 \times f_3 \right\} = \left(-1\right)^{f_1+f_2+f_3+f_4+f_5+f_6+f_7+f_8} \left\{ h_1 \times h_2 \right\} \left\{ h_3 \times h_4 \right\} \left\{ h_5 \times h_6 \right\} \left\{ h_7 \times h_8 \right\} 
\]

(6.54)

See also (24.10).

\[
\sum_{s}(s) \left\{ f_1 \ldots f_{n-1} \right\} \times \left[ \begin{array}{cccc}
 f_1 & f_2 & \cdots & f_{n-1} \\
 h_1 & h_2 & \cdots & h_{n-1} \\
 & & \ddots & \vdots \\
 & & & h_1
\end{array} \right] 
\]

(6.55)
Appendix 7

The simplest summation and transformation formulas
for \(jm\)-coefficients

In this appendix formulas are given only for those \(jm\)-coefficients the diagrams of which contain no more than one closed cycle.

\[
\sum (j_s j_t j_f) (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f} = \\
= \sum_{m_{-m}} (-1)^{i_{-m} + i_{+m}} \frac{J_s J_t J_f}{m_s m_t m_f} = \\
= \sum_{m_{-m}} (-1)^{i_{-m} - m'} \frac{J_s J_t J_f}{m_s m_t m_f} = \\
= \sum_{m_{-m}} (-1)^{i_{-m} - m'} \frac{J_s J_t J_f}{m_s m_t m_f} \times (j_s j_t j_f) \times (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f}.
\] (A. 7.1)

\[
\sum_{i_{-m}} (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f} = (j_s j_t j_f) \times (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f}.
\] (A. 7.2)

\[
\sum_{i_{-m}} (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f} = (j_s j_t j_f) \times (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f}.
\] (A. 7.3)

\[
\sum_{i_{-m}} (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f} = (j_s j_t j_f) \times (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f}.
\] (A. 7.4)

\[
\sum_{i_{-m}} (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f} = (j_s j_t j_f) \times (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f}.
\] (A. 7.5)

\[
\sum_{i_{-m}} (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f} = (j_s j_t j_f) \times (-1)^{i_{-m}} \frac{J_s J_t J_f}{m_s m_t m_f}.
\] (A. 7.6)