CTC2, NWI-MOL176, exercises week 3,

Gerrit C. Groenenboom, 23-April-2025

Question 1: Questions chapter 5 (II)

1a. In the derivation in Chapter 5.8 we used

$$(\boldsymbol{n} \times \boldsymbol{r}) \cdot \boldsymbol{\nabla} = \boldsymbol{n} \cdot (\boldsymbol{r} \times \boldsymbol{\nabla}). \tag{1}$$

Derive this equation using the Levi-Civita tensor.

1b. Show that the Wigner $\boldsymbol{D}$-matrix for rotation around the y-axis is real (see Section 5.11, Eq. (5.119) of the lecture notes).

1c. Compute the matrix elements of the rotation operator

$$\langle lm|\hat{R}(\boldsymbol{e}_z, \alpha)|lm'\rangle. \tag{2}$$

1d. Compute the Wigner D-matrix elements

$$d^{(l)}_{mk}(\beta) = \langle lm|e^{-\frac{i}{\hbar}\beta \hat{l}_y}|lk\rangle$$

for $l = 1/2$.

1e. Show that the Wigner-D matrices satisfy the matrix representation property

$$\boldsymbol{D}^{(l)}(\hat{R}_1\hat{R}_2) = \boldsymbol{D}^{(l)}(\hat{R}_1)\boldsymbol{D}^{(l)}(\hat{R}_2), \tag{3}$$

starting from the defining equation of the $\boldsymbol{D}$-matrices.

Question 2: Questions chapter 7

For Jacobi vectors $\boldsymbol{r}$ and $\boldsymbol{R}$ the corresponding angular momentum operators are defined by

$$\hat{\boldsymbol{j}} = \boldsymbol{r} \times \hat{\boldsymbol{p}}_r \tag{4}$$

$$\hat{\boldsymbol{l}} = \boldsymbol{R} \times \hat{\boldsymbol{p}}_R, \tag{5}$$

where $\hat{\boldsymbol{p}}_r$ and $\hat{\boldsymbol{p}}_R$ are the momentum operators for $\boldsymbol{r}$ and $\boldsymbol{R}$, respectively. The total angular momentum operator is defined by

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{j}} + \hat{\boldsymbol{l}}. \tag{6}$$

2a. Derive the following commutation relations, for $i = x, y, z$,

$$[\hat{\boldsymbol{J}}_i, r^2] = 0 \tag{7}$$

$$[\hat{\boldsymbol{J}}_i, R^2] = 0 \tag{8}$$

$$[\hat{\boldsymbol{J}}_i, \boldsymbol{r} \cdot \boldsymbol{R}] = 0. \tag{9}$$

2b. For two Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ that commute, $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{0}$, show that

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}e^{\boldsymbol{B}}. \tag{10}$$

2c. Show that the previous result also holds if the matrices are not Hermitian (but still commute).