CTC2, NWI-MOL176, exercises week 2

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Question 1: Questions chapter 5

In section 5.1 of the lecture notes, the angular momentum states $|ab\rangle$ are defined by

$$\hat{l}^2|ab\rangle = a\hbar^2|ab\rangle \tag{1}$$

$$\hat{l}_z|ab\rangle = b\hbar|ab\rangle. \tag{2}$$

Ladder operators are defined by

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y. \tag{3}$$

1a. Show that $b^2 \leq a$, using Eqs. (5.1)-(5.4) of the lecture notes and the properties of scalar products (section 1.5.3 of the lecture notes)

1b. The angular momentum ladder operators are each other's Hermitian conjugates, $\hat{l}_\pm^\dagger = \hat{l}_\mp$. Derive this result using the definition of Hermitian conjugate and the defining properties of scalar products.

1c. Show that

$$\hat{l}_\pm\hat{l}_\mp = \hat{l}^2 - \hat{l}_z^2 \pm \hbar\hat{l}_z.$$

1d. Compute the matrix representation of $\hat{l}^2$, $\hat{l}_z$, $\hat{l}_+$, $\hat{l}_-$, $\hat{l}_x$, and $\hat{l}_y$ in the angular momentum basis

$$\{|lm\rangle, m = -l, -l+1, \ldots, l\}, \quad \text{with } l = 1.$$

1e. Check the relation $\hat{l}^2 = \hat{l}_z^2 + \hbar\hat{l}_z + \hat{l}_-\hat{l}_+$ for the $l = 1$ matrix representations from the previous question.

1f. Compute

$$\boldsymbol{B} = \cos(\boldsymbol{A}),$$

for the 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hint: first solve the eigenvalue problem, Eq. (5.35) in the lecture notes and then use Eq. (5.47) with $f(\lambda_i) = \cos(\lambda_i)$.

According to section 5.6 of the lecture notes, the translation operator [Eq. (5.54)] acting on $\sin(x)$ gives [Eq. (5.49)]

$$e^{-\alpha\frac{\partial}{\partial x}}\sin(x) = \sin(x - \alpha).$$

1g. Show that this equation is correct in first order, i.e., for small α.

1h. Compute

$$\left(e^{-3\frac{\partial}{\partial x}}\right)^2 e^{-x^2}.$$

1i. What is wrong in this derivation:

$$e^{\frac{\partial}{\partial x}}e^x = e^{\frac{\partial}{\partial x}x} = e^1 = e\ ?$$

1j. Repeat question **1f**, but now for the complex matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The angular momentum operator $\hat{l}_z$ in spherical polar coordinates, is given by [see lecture notes Eq. (5.66)]

$$\hat{l}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{4}$$

1k. Derive this result starting from the expression for $\hat{l}_z$ in Cartesian coordinates.