CTC2, NWI-MOL176, exercises week 2,

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Question 1: Questions chapter 5

In section 5.1 of the lecture notes, the angular momentum states $|ab\rangle$ are defined by

$$\hat{l}^2|ab\rangle = a\hbar^2|ab\rangle \tag{1}$$

$$\hat{l}_z|ab\rangle = b\hbar|ab\rangle. \tag{2}$$

Ladder operators are defined by

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y. \tag{3}$$

1a. Show that $b^2 \leq a$, using Eqs. (5.1)-(5.4) of the lecture notes and the properties of scalar products (section 1.5.3 of the lecture notes)

Answer: *The expectation value of the square of a Hermitian operator must be positive, e.g.,*

$$\langle ab|\hat{l}_x^2|ab\rangle = \langle \hat{l}_x ab||\hat{l}_x ab\rangle = |\hat{l}_x|ab\rangle|^2 \geq 0. \tag{4}$$

Since

$$\langle ab|\hat{l}^2|ab\rangle - \langle ab|\hat{l}_z^2|ab\rangle = \langle ab|\hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2|ab\rangle - \langle ab|\hat{l}_z^2|ab\rangle = \langle ab|\hat{l}_x^2|ab\rangle + \langle ab|\hat{l}_y^2|ab\rangle \geq 0 \tag{5}$$

$$a\hbar^2 - (b\hbar)^2 \geq 0 \tag{6}$$

$$a \geq b^2. \tag{7}$$

1b. The angular momentum ladder operators are each other's Hermitian conjugates, $\hat{l}_\pm^\dagger = \hat{l}_\mp$. Derive this result using the definition of Hermitian conjugate and the defining properties of scalar products.

Answer:

$$\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y. \tag{8}$$

Since $\hat{l}_x$ is Hermitian we have $\hat{l}_x^\dagger = \hat{l}_x$. Thus, we have to show

$$(i\hat{l}_y)^\dagger = -\hat{l}_y. \tag{9}$$

Since $\hat{l}_y$ is Hermitian, for any state $s|\phi\rangle$, χ we have

$$\langle \phi|\hat{l}_y\chi\rangle = \langle \hat{l}_y\phi|\chi\rangle \tag{10}$$

Thus, for these states we also have,

$$\langle \phi|i\hat{l}_y\chi\rangle = i\langle \hat{l}_y\phi|\chi\rangle = \langle -i\hat{l}_y\phi|\chi\rangle, \tag{11}$$

and hence $(i\hat{l}_y)^\dagger = -i\hat{l}_y$, q.e.d.

1c. Show that

$$\hat{l}_\pm\hat{l}_\mp = \hat{l}^2 - \hat{l}_z^2 \pm \hbar\hat{l}_z.$$

Answer:

$$\hat{l}_\pm\hat{l}_\mp = (\hat{l}_x \pm i\hat{l}_y)(\hat{l}_x \mp i\hat{l}_y) \tag{12}$$

$$= \hat{l}_x^2 + \hat{l}_y^2 \mp i[\hat{l}_x, \hat{l}_y] \tag{13}$$

$$= \hat{l}^2 - \hat{l}_z^2 \mp i^2\hbar\hat{l}_z \tag{14}$$

$$= \hat{l}^2 - \hat{l}_z^2 \pm \hbar\hat{l}_z. \tag{15}$$

1d. Compute the matrix representation of $\hat{l}^2$, $\hat{l}_z$, $\hat{l}_+$, $\hat{l}_-$, $\hat{l}_x$, and $\hat{l}_y$ in the angular momentum basis

$$\{|lm\rangle, m = -l, -l+1, \ldots, l\}, \quad \text{with } l = 1.$$

Answer:

$$\boldsymbol{L}^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\boldsymbol{L}_z = \hbar \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{L}_+ = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\boldsymbol{L}_- = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

1e. Check the relation $\hat{l}^2 = \hat{l}_z^2 + \hbar\hat{l}_z + \hat{l}_-\hat{l}_+$ for the $l = 1$ matrix representations from the previous question.

Answer:

$$\boldsymbol{L}_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

$$\boldsymbol{L}_-\boldsymbol{L}_+ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Adding these matrices, together with the $\boldsymbol{L}_z$ above gives indeed $\boldsymbol{L}^2$.

1f. Compute

$$\boldsymbol{B} = \cos(\boldsymbol{A}),$$

for the 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hint: first solve the eigenvalue problem, Eq. (5.35) in the lecture notes and then use Eq. (5.47) with $f(\lambda_i) = \cos(\lambda_i)$.

Answer:

$$\det(\boldsymbol{A} - \lambda\boldsymbol{1}_{2\times2}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0, \quad \textit{so } \lambda = \pm 1.$$

Solving, for $\lambda = 1$,

$$(\boldsymbol{A} - \boldsymbol{1}_{2\times2})\boldsymbol{u}_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} U_{1,1} \\ U_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{18}$$

gives

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

where the solution is normalized to one. For $\lambda = -1$ we find the eigenvector

$$\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{20}$$

so the matrix of eigenvectors is

$$\boldsymbol{U} = (\boldsymbol{u}_1 \boldsymbol{u}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{21}$$

and the diagonal matrix with eigenvectors is

$$\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

and so

$$\boldsymbol{B} = \cos \boldsymbol{A} = \boldsymbol{U} \begin{pmatrix} \cos(1) & 0 \\ 0 & \cos(-1) \end{pmatrix} \boldsymbol{U}^\dagger \tag{23}$$

We get

$$\boldsymbol{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \cos(1) & 0 \\ 0 & \cos(-1) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \cos(1) \mathbf{1}_{2\times 2}, \tag{24}$$

where we used $\cos(-1) = \cos(1)$.

According to section 5.6 of the lecture notes, the translation operator [Eq. (5.54)] acting on $\sin(x)$ gives [Eq. (5.49)]

$$e^{-\alpha \frac{\partial}{\partial x}} \sin(x) = \sin(x - \alpha).$$

1g. Show that this equation is correct in first order, i.e., for small α.

Answer: *For small α*

$$e^{-\alpha \frac{\partial}{\partial x}} \approx 1 - \alpha \frac{\partial}{\partial x} \tag{25}$$

so

$$e^{-\alpha \frac{\partial}{\partial x}} \sin(x) \approx \sin(x) - \alpha \frac{\partial}{\partial x} \sin(x) \tag{26}$$

which is indeed the Taylor series of $\sin(x)$ to first order.

1h. Compute

$$\left(e^{-3\frac{\partial}{\partial x}}\right)^2 e^{-x^2}.$$

Answer: *With*

$$e^{-3\frac{\partial}{\partial x}} f(x) = f(x-3) \tag{27}$$

we get

$$(e^{-3\frac{\partial}{\partial x}})^2 e^{-x^2} = e^{-3\frac{\partial}{\partial x}} e^{-3\frac{\partial}{\partial x}} e^{-x^2} = e^{-(x-6)^2}. \tag{28}$$

1i. What is wrong in this derivation:

$$e^{\frac{\partial}{\partial x}} e^x = e^{\frac{\partial}{\partial x} x} = e^1 = e\ ?$$

Answer: *For numbers a and b we have*

$$e^a e^b = e^{a+b}, \tag{29}$$

i.e., not e^{ab}. Furthermore, for operators $\hat{a}$ and $\hat{b}$ this relation only holds if the commutator $[\hat{a}, \hat{b}] = 0$, and here we have $[\frac{\partial}{\partial x}, x] = 1 \neq 0$.

We can also explicitly calculate the result, noticing that the first exponential is a translation operator *of* -1,

$$e^{\frac{\partial}{\partial x}} e^{x} = e^{x+1}. \tag{30}$$

1j. Repeat question **1f**, but now for the complex matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Answer: *We now have*

$$|\boldsymbol{A} - \lambda \mathbf{1}_{2\times 2}| = \begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0, \quad \lambda = \pm 1 \tag{31}$$

as before. For the eigenvectors, however, we find

$$\boldsymbol{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \tag{32}$$

still,

$$\boldsymbol{B} = \cos \boldsymbol{A} = \boldsymbol{U} \cos(\boldsymbol{\Lambda}) \boldsymbol{U}^{\dagger} = \cos(1) \mathbf{1}_{2\times 2}. \tag{33}$$

The angular momentum operator $\hat{l}_z$ in spherical polar coordinates, is given by [see lecture notes Eq. (5.66)]

$$\hat{l}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}. \tag{34}$$

1k. Derive this result starting from the expression for $\hat{l}_z$ in Cartesian coordinates.

Answer: *From the definition:*

$$\hat{l}_z = x \hat{p}_y - y \hat{p}_x = \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \tag{35}$$

With

$$x = r \cos\phi \sin\theta \tag{36}$$
$$y = r \sin\phi \sin\theta \tag{37}$$
$$z = r \cos\theta \tag{38}$$

and starting from Eq. (34) we find

$$\hat{l}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \tag{39}$$
$$= \frac{\hbar}{i} \left(\frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} + \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial z}{\partial \phi} \frac{\partial}{\partial z} \right) \tag{40}$$
$$= \frac{\hbar}{i} \left(r \cos\phi \sin\theta \frac{\partial}{\partial y} - r \sin\phi \sin\theta \frac{\partial}{\partial x} + 0 \right) \tag{41}$$
$$= \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right), \tag{42}$$

q.e.d.