Excercises Angular Momentum Theory

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Excercise 1: Rotating Legendre polynomials

Legendre polynomials are related to Racah normalized spherical harmonics through

$$P_l(\cos\theta) = C_{l,0}(\theta, 0) = C_{l,0}(\theta, \phi).$$

A rotation operator for rotation over zyz Euler angles (α, β, γ) is denoted as $\hat{R}(\alpha, \beta, \gamma)$

1a. Express

$$\hat{R}(\alpha, \beta, \gamma) C_{lm}(\theta, \phi)$$

as a linear comination of Racah normalized spherical harmonics

1b. What is wrong with the expression

$$\hat{R}(\alpha, \beta, \gamma) C_{l,0}(\theta, 0) = \hat{R}(\alpha, \beta, \gamma) C_{l,0}(\theta, \phi)?$$

Excercise 2: Symmetry of Clebsch-Gordan coefficients

Clebsch-Gordan coefficients are related to $3 - j$ symbols through

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1 - j_2 + m_3} \sqrt{2j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}.$$

Permuting any two columns in the $3-j$ symbol gives a phase factor $(-1)^{j_1+j_2+j_3}$, e.g.

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

2a. Derive the phase factor F_{12} in the expression

$$\langle j_2 m_2 j_1 m_1 | j_3 m_3 \rangle = F_{12} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$$

2b. Derive the phase factor F_{23} in the expression

$$\langle j_1 m_1 j_3 m_3 | j_2 m_2 \rangle = F_{23} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$$

Excercise 3: Spherical components of Cartesian vectors

The spherical components $\boldsymbol{x} = [s_{+1}, s_0, s_{-1}]^T$ of the vector Cartesian vector $\boldsymbol{x} = [x_1, x_2, x_3]^T$ are given by

$$\begin{aligned} s_{+1} &= -\frac{x_1 + ix_2}{\sqrt{2}} \\ s_0 &= x_3 \\ s_{-1} &= \frac{x_1 - ix_2}{\sqrt{2}}. \end{aligned}$$

To rotate a vector given by its spherical components one may use Wigner rotation matrixes of rank 1

$$\boldsymbol{s}' = \boldsymbol{D}^{(1)*}(\alpha, \beta, \gamma)\boldsymbol{s}$$

3a. Derive this expression.

3b. Give the spherical components of the Cartesian vector $[1, 0, 0]^T$.

3c. Use the given expression to rotate this vector around the z-axis over an angle γ and transform the spherical components of the rotated vector back to Cartesian components.

3d. Give the transformation matrix $\boldsymbol{S}$ defined by

$$\boldsymbol{s} = \boldsymbol{S}\boldsymbol{x}$$

3e. Show that $\boldsymbol{S}$ is unitary and give the inverse of $\boldsymbol{S}$.

3f. The scalar product of two vectors is invariant under rotations. Derive an expression for the scalar product of two vectors as a coupling of its spherical components to a scalar invariant using Clebsch-Gordan coefficients. The normalization constant is derived below.

3g. Use the recursion relations in the lecture notes to derive the value of $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$ for $j_1 = j_2 = 1$ and $j_3 = 0$ and all possible values of m_1, m_2, and m_3 and determine the normalization constant.

3h. Use the recursion relations in the lecture notes to derive the value of $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$ for $j_1 = j_2 = j_3 = 1$ and all possible values of m_1, m_2, and m_3.

3i. Determine the expression for the spherical components of the cross product of two vectors with spherical components $\boldsymbol{s}$ and $\boldsymbol{t}$.