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①

$$f_{jm}(x,y) \equiv \frac{x^{j+m}\, y^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}} \qquad \begin{cases} 0 \le j \\ -j \le m \le j \end{cases}$$

Gerrit C. Groenenboom

definiëer lineaire ruimte met basis $\{ f_{jm},\ j=0,1/2,1,1\tfrac{1}{2},\ldots;\ m=-j,-j+1,\ldots,j \}$

inproduct

$$\cancel{\langle f | g \rangle = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left[\frac{\partial^{p+q}}{\partial x^p \partial y^q} f(x,y)\right]_{x,y=0} \left[\frac{\partial^{p+q}}{\partial x^p \partial y^q} g(x,y)\right]_{(x,y)=0}}$$

$$\langle j m | j' m' \rangle \equiv \delta_{jj'}\, \delta_{mm'}$$

SYMMETRY of $d^{j}_{m'm}(\beta)$

operatoren

$$\hat{j}_x \equiv \tfrac{1}{2}\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)$$

$$\hat{j}_y \equiv \tfrac{1}{2i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\hat{j}_z \equiv \tfrac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

$$[\hat{j}_x, \hat{j}_y] = \frac{1}{4i}\left[x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

$$\overset{(a)\,(b)\,(c)\,(d)}{=} \frac{1}{4i}\left\{ -\left[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] + \left[y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right]\right\}$$

$$= \frac{1}{4i}\left\{ -\left(y\left[x,\frac{\partial}{\partial x}\right]\frac{\partial}{\partial y} + x\left[\frac{\partial}{\partial y}, y\right]\frac{\partial}{\partial x}\right) + x\left[y,\frac{\partial}{\partial y}\right]\frac{\partial}{\partial x} + y\left[\frac{\partial}{\partial x}, x\right]\frac{\partial}{\partial y}\right\}$$

$$= \frac{1}{4i}\left\{ 2y\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial x}\right\} = i\cdot\tfrac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) = i\hat{j}_z$$

$$[\hat{j}_y, \hat{j}_z] = \frac{1}{4i}\left\{\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right]\right\}$$

$$= \frac{1}{4i}\left\{\left[x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}\right] - \left[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\right] - \left[y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}\right] + \left[y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right]\right\}$$

$$= \frac{1}{4i}\left\{ x\left[x,\frac{\partial}{\partial x}\right]\frac{\partial}{\partial y} - x\left[\frac{\partial}{\partial y}, y\right]\frac{\partial}{\partial y} - y\left[\frac{\partial}{\partial x}, x\right]\frac{\partial}{\partial x} + y\left[y,\frac{\partial}{\partial y}\right]\frac{\partial}{\partial x}\right\}$$

$$= \frac{1}{4i}\left\{ -x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}\right\} = i\cdot\tfrac{1}{2}\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right) = i\hat{j}_x$$

$$[\hat{j}_z, \hat{j}_x] = \frac{1}{4}\left\{\left[x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},\ x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right]\right\}$$

$$= \frac{1}{4}\left\{\left[x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] + \left[x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}\right] - \left[y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}\right] - \left[y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right]\right\}$$

$$= \frac{1}{4}\left\{ x\left[\frac{\partial}{\partial x}, x\right]\frac{\partial}{\partial y} + y\left[x,\frac{\partial}{\partial x}\right]\frac{\partial}{\partial x} - x\left[y,\frac{\partial}{\partial y}\right]\frac{\partial}{\partial y} - y\left[\frac{\partial}{\partial y}, y\right]\frac{\partial}{\partial x}\right\}$$

$$= \frac{1}{4}\left\{ x\frac{\partial}{\partial y} + -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right\} = i\,\frac{1}{2i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = i\hat{j}_y$$

$$\hat{j}_+ \equiv \hat{j}_x + i\hat{j}_y = x\frac{\partial}{\partial y}$$

$$\hat{j}_- \equiv \hat{j}_x - i\hat{j}_y = y\frac{\partial}{\partial x}$$

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②

$$\hat{J}_z f_{j,m}(x,y) = \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\frac{x^{j+m}\,y^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= \frac{1}{2}\left((j+m) - (j-m)\right)\frac{x^{j+m}\,y^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= m\, f_{j,m}(x,y)$$

$$\hat{J}_+ f_{j,m}(x,y) = x\frac{\partial}{\partial y}\frac{x^{j+m}\,y^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= (j-m)\frac{x^{j+m+1}\,y^{j-m-1}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= \sqrt{(j+m+1)(j-m)}\;\frac{x^{j+m+1}\,y^{j-(m+1)}}{[(j+m+1)!\,(j-(m+1))!]^{1/2}}$$

$$= c^+_{j,m}\, f_{j,m+1}(x,y)$$

$\underline{opm}$ $c^+_{j,j} = 0$ $(f_{j,j+1} \equiv 0)$

$\left[(j+m+1)(j-m) = j^2 + j - m(m+1) - jm + jm = j(j+1) - m(m+1)\right]$

$$\hat{J}_- f_{j,m}(x,y) = y\frac{\partial}{\partial x}\frac{x^{j+m}\,y^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= (j+m)\frac{x^{j+m-1}\,y^{j-(m-1)}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= \sqrt{(j+m)(j-m+1)}\;\frac{x^{j+m-1}\,y^{j-(m-1)}}{[(j+m-1)!\,(j-(m-1))!]^{1/2}}$$

$$= c^-_{j,m}\, f_{j,m-1}(x,y)$$

$\left[(j+m)(j-m+1) = j^2 - m^2 + j + m = j(j+1) - m(m-1)\right]$

$$J^2 = J_x^2 + J_y^2 + J_z^2 = \cancel{\ldots}\; J_+ J_- - J_z + J_z^2$$

$$J^2 |j,m\rangle = \left(c^+_{j,m-1}\, c^-_{j,m} - m + m^2\right)|j\,m\rangle$$

$$= \left[\sqrt{j(j+1) - m(m-1)}\sqrt{j(j+1) - m(m-1)} - m + m^2\right]|j\,m\rangle = j(j+1)\,|j\,m\rangle$$

$j = 1/2$, basis $\{|1/2\ 1/2\rangle, |1/2, -1/2\rangle\}$

$$\langle 1/2\ 1/2 | J_+ | 1/2, -1/2\rangle = c^+_{1/2,-1/2} = \sqrt{1/2(1/2+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}+1)} = 1 \quad ; \quad J_+ = \begin{array}{c|cc|c} & 1/2 & -1/2 & \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & & \begin{array}{c} 1/2 \\ -1/2 \end{array}\end{array}$$

$$\langle 1/2\ -1/2 | J_- | 1/2, 1/2\rangle = c^-_{1/2,1/2} = \sqrt{1/2(1/2+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)} = 1 \quad ; \quad J_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_y = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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$$D^{(j)}(\alpha,\beta,\gamma) \equiv e^{-i\alpha J_z^{(j)}}\, e^{-i\beta J_y^{(j)}}\, e^{-i\gamma J_z^{(j)}}$$

$$\langle j m' | D(\alpha\beta\gamma) | j m\rangle = e^{-i\alpha m'}\, d^{j}_{m'm}(\beta)\, e^{-i\gamma m}$$

$$d^{(j)} = e^{-i\beta J_y^{(j)}}$$

$$d^{(1/2)} = e^{M} \; ; \quad M = -i\beta \tfrac{1}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix} = \tfrac{\beta}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}$$

$$M^2 = -\left(\tfrac{\beta}{2}\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} \;\rightarrow\; M^{2n} = (-1)^n \left(\tfrac{\beta}{2}\right)^{2n}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$$

$$M^{2n+1} = \tfrac{\beta}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}(-1)^n\left(\tfrac{\beta}{2}\right)^{2n}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$$

$$= (-1)^n\left(\tfrac{\beta}{2}\right)^{2n+1}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}$$

$$e^M = \sum_{n=0}^{\infty}\tfrac{1}{n!}M^n = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{\beta}{2}\right)^{2n} + \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{\beta}{2}\right)^{2n+1}$$

$$= \begin{array}{cc} m'=1/2 & m=-1/2 & \\ \left[\begin{array}{cc} \cos\frac{\beta}{2} & -\sin\beta/2 \\ \sin\frac{\beta}{2} & \cos\beta/2 \end{array}\right] & & \begin{array}{l} m'=1/2 \\ m'=-1/2\end{array}\end{array}$$

$$\hat{J}_y(f\cdot g) = \tfrac{1}{2i}\left(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\right)(f\cdot g)$$

$$= \left\{\left(\tfrac{1}{2i}\, x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\right)f\right\} g + f\left\{\tfrac{1}{2i}\left(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\right)g\right\}$$

$$= (\hat{J}_y f)\cdot g + f(\hat{J}_y g)$$

$$\hat{J}_y^{\,n}(f\cdot g) = \sum_{k=0}^{n}\binom{n}{k}(J_y^k\cdot f)(J_y^{n-k} g)$$

$$\hat{A} = -\beta \hat{J}_y \;\Rightarrow\; A^n(f\cdot g) = \sum_{k=0}^{n}\binom{n}{k}(\hat{A}^k\cdot f)(A^{n-k} g)$$

$$e^{\hat{A}} f\cdot g = \sum_{n=0}^{\infty}\tfrac{1}{n!}A^n(f\cdot g) = \sum_{n=0}^{\infty}\tfrac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(\hat{A}^k f)\cdot\hat{A}^{n-k} g$$

$$(e^{\hat{A}} f)(e^{\hat{A}} g) = \left\{\sum_{i=0}^{\infty}\tfrac{1}{i!}A^i f\right\}\left\{\sum_{j=0}^{\infty}\tfrac{1}{j!}A^j g\right\}$$

$$n \equiv i+j$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\tfrac{1}{i!}A^i f\right\}\tfrac{1}{(n-i)!}A^{(n-i)} g$$

$$= \sum_{n=0}^{\infty}\tfrac{1}{n!}\sum_{k=0}^{n}\tfrac{n!}{k!(n-k)!}(A^k f)(A^{n-k} g) = e^{\hat{A}}(f\cdot g)$$

$$e^{A} f\cdot g\cdot h = (e^{A} f\cdot g)\, e^{A}\cdot h = (e^{A} f)(e^{A} g)(e^{A} h) \quad \underline{\text{etc}}$$

$$|\tfrac12\ \tfrac12\rangle = \frac{x^{1/2+1/2}\ y^{1/2-1/2}}{[(1/2+1/2)!\ (1/2-1/2)!]^{1/2}} = x$$

$$|\tfrac12\ -\tfrac12\rangle = y$$

$$\begin{cases} e^{-i\beta \hat{J}_y}\, x = \cos\frac{\beta}{2}\, x + \sin\frac{\beta}{2}\cdot y \\ e^{-i\beta J_y}\, y = -\sin\frac{\beta}{2}\, x + \cos\frac{\beta}{2}\, y \end{cases}$$

$$D(0\beta 0)\,|jm\rangle = \sum_{m'} |jm'\rangle\langle jm'|\,D(0\beta 0)|jm\rangle$$
$$= \sum_{m'} |jm'\rangle\, D^j_{m'm}(0,\beta,0)$$
$$= \sum_{m'} |jm'\rangle\, d^j_{m'm}(\beta)$$

$$e^{-i\beta J_y} f_{jm}(x,y) = e^{-i\beta J_y} \frac{x^{j+m}\, y^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= \frac{\left\{e^{-i\beta J_y} x\right\}^{j+m}\left\{e^{-i\beta J_y} y\right\}^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$= \frac{\left(\cos\frac{\beta}{2}x + \sin\frac{\beta}{2}y\right)^{j+m}\left(-\sin\frac{\beta}{2}x + \cos\frac{\beta}{2}y\right)^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$$\equiv \sum_{m'} \frac{x^{j+m'}\, y^{j-m'}}{[(j+m')!\,(j-m')!]^{1/2}}\, d^j_{m'm}(\beta)$$

Symmetry

$$D(0,\pi,0)\,|jm\rangle = (-1)^{j-m} \frac{y^{j+m}\, x^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}} = (-1)^{j-m}\,|j,-m\rangle$$

$$d^j_{m'm}(\pi) = (-1)^{j-m}\langle jm'|j-m\rangle = (-1)^{j-m}\,\delta_{m',-m} = (-1)^{j+m'}\,\delta_{m',-m}$$

$$D(0,-\pi,0)\,|jm\rangle = (-1)^{j+m}\frac{y^{j+m}\, x^{j-m}}{[(j+m)!\,(j-m)!]} = (-1)^{j+m}\,|j,-m\rangle$$

$$d^j_{m'm}(-\pi) = (-1)^{j+m}\,\delta_{m',-m}$$

Unitary + real $\rightarrow$ orthonormal $\qquad R^{-1} = R^T$

$$d^j_{m'm}(-\beta) = d^j_{mm'}(\beta)$$

$$d^j_{m'm}(\pi) = (-1)^{j-m}\delta_{m',-m}$$

$$d^j_{m'm}(-\pi) = (-1)^{j+m}\delta_{m',-m}$$

$$d^j_{m'm}(0) = \sum_\mu d^j_{m'\mu}(\pi)\, d^j_{\mu m}(-\pi)$$
$$= \sum_\mu (-1)^{j-\mu}\delta_{m',-\mu}\,(-1)^{j+m}\delta_{\mu,-m}$$
$$= (-1)^{2(j+m)}\delta_{m',m} = \delta_{m',m}$$

$$d^j_{m'm}(\pi-\beta) = \sum_\mu d^j_{m'\mu}(\pi)\, d_{\mu m}(-\beta)$$
$$= \sum_\mu (-1)^{j-\mu}\delta_{m',-\mu}\, d_{m,\mu}(\beta)$$
$$= (-1)^{j+m'} d^j_{m,-m'}(\beta)$$

$$d^j_{m'm}(\pi+\beta) = \sum_\mu d^j_{m'\mu}(\pi)\, d_{\mu m}(\beta)$$
$$= \sum_\mu (-1)^{j-\mu}\delta_{m',-\mu}\, d_{\mu m}(\beta) = (-1)^{j+m'} d_{-m',m}(\beta)$$

$$d^j_{m'm}(\beta) = (-1)^{j+m'} d^j_{m,-m'}(\pi-\beta)$$
$$= (-1)^{j+m'}(-1)^{j+m} d_{-m',m}(\beta)$$
$$= (-1)^{2j+m+m'} d_{-m',-m}(\beta)$$
$$= (-1)^{m'-m} d_{-m',-m}(\beta)$$

PF: $d^{(j)}_{m'm}(\beta) = d^{(j)}_{-m,-m'}(\beta) = (-1)^{m'-m} d^{(j)}_{m,m'}(\beta)$

$\underline{d^j_{m'm}(\pi+\beta) = (-1)^{j+m'} d_{-m',m}(\beta)} \Rightarrow d_{-m',m}(\beta) = (-1)^{j+m'} d_{m'm}(\pi+\beta)$

$d_{m'm}(\pi-\beta) = (-1)^{j+m'} d_{m,-m'}(\beta)$ (Zare 3.75) $\downarrow(3.67)$

$= (-1)^{j+m'} d_{m',-m}(\beta)$ ~~(Zare 3.75)~~

$= (-1)^{j+m'}(-1)^{m'+m} d_{-m',m}(\beta)$

$d_{m',m}(\beta) = (-1)^{j-m'} d_{-m',m}(\pi+\beta)$

$= (-1)^{j-m'}(-1)^{-m'-m} d_{m',-m}(\pi+\beta)$

$= (-1)^{j+m} d_{m',-m}(\pi+\beta)$

$$\boxed{d_{m'm}(\pi-\beta) = (-1)^{j-m} d_{-m',m}(\beta)}\ \boxed{= (-1)^{j+m'} d_{m',-m}(\beta)}$$

$$e^{-i\beta J_y}|jm\rangle = \frac{\left(\cos\frac{\beta}{2}\, x + \sin\frac{\beta}{2}\, y\right)^{j+m}\left(-\sin\frac{\beta}{2}\, x + \cos\frac{\beta}{2}\, y\right)^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

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$$= \sum_{m'} \frac{x^{j+m'}\, y^{j-m'}}{[(j+m')!\,(j-m')!]^{1/2}}\, d^{j}_{m'm}(\beta)$$

$$\sum_{k=0}^{j+m}\binom{j+m}{k}\left(\cos\frac{\beta}{2}\, x\right)^{j+m-k}\left(\sin\frac{\beta}{2}\, y\right)^{k}$$

$$\times \sum_{l=0}^{j-m}\binom{j-m}{l}\left(-\sin\frac{\beta}{2}\, x\right)^{l}\left(\cos\frac{\beta}{2}\, y\right)^{j-m-l}$$

$$= \sum_{m'} \frac{x^{j+m'}\, y^{j-m'}}{[(j+m')!\,(j-m')!]^{1/2}}\, d^{j}_{m'm}(\beta)\,[(j+m)!\,(j-m)!]^{1/2}$$

$$j+m-k+l = j+m'$$

$$l = m'-m+k$$

$$j+m-k+j-m-m'+m-k = 2j+m-m'-2k$$

$$\sum_{k=0}^{j+m}\binom{j+m}{k}\left(\cos\frac{\beta}{2}\right)^{j+m-k}\left(\sin\frac{\beta}{2}\right)^{m'-m+k}(-1)^{m'-m+k}\left(\sin\frac{\beta}{2}\right)^{k}\left(\cos\frac{\beta}{2}\right)^{j-m-(m'-m+k)}$$

$$\binom{j-m}{m'-m+k} \times x^{j+m-k+m'-m+k}\, y^{k+j-m-(m'-m+k)}$$

$$= \sum_{k=0}^{j+m}\binom{j+m}{k}\binom{j-m}{m'-m+k}(-1)^{m'-m+k}\left(\sin\frac{\beta}{2}\right)^{m'-m+2k}\left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2k} x^{j+m'}\, y^{j-m'}$$

$$= \sum_{m'} \frac{x^{j+m'}\, y^{j-m'}}{[(j+m')!\,(j-m')!]^{1/2}}\, d^{j}_{m'm}(\beta)\,[(j+m)!\,(j-m)!]^{1/2}$$

$$d^{j}_{m'm}(\beta) = \left[\frac{(j+m')!\,(j-m')!}{(j+m)!\,(j-m)!}\right]^{1/2}\sum_{k=0}^{j+m}\binom{j+m}{k}\binom{j-m}{m'-m+k}(-1)^{m'-m+k}\left(\sin\frac{\beta}{2}\right)^{m'-m+2k}\left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2k}$$

$$= [(j+m')!\,(j-m)!\,(j-m')!\,(j+m)!]^{1/2}\sum_{k=0}^{j+m}\frac{(-1)^{m'-m+k}\left(\sin\frac{\beta}{2}\right)^{m'-m+2k}\left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2k}}{k!\,(j+m-k)!\,(m'-m+k)!\,(j-m'-k)!}$$

dus $d^{j}_{m'm}(\beta) = d^{j}_{-m,-m'}(\beta)$
$$\left\{\begin{array}{l} m \to -m' \\ m' \to -m \\ m-m' = m-m' \end{array}\right.$$

Alternative derivation of symmetry relation

$$d_{m'm}(\beta) = (-1)^{m-m'}\, d_{mm'}(\beta)$$

$$D(\cancel{R_z^T(\pi)\, R(\alpha,\beta,\gamma)\, R_z(\pi)})$$

$$R_z^T(\pi)\; R_z(\alpha) R_y(\beta) R_z(\gamma)\; R_z(\pi) = R_z(\alpha+\pi)\, R_y(\beta)\, R_z(\gamma+\pi)$$

$$= R_z(\alpha)\, R_z(\pi)\;\; R_y(\beta)\;\; R_z(\pi)\; R_z(\gamma)$$

$$= R_z(\alpha)\;\; R_z(-\pi)\;\; R_z(\pi)\; R_y(-\beta)\; R_z(\gamma) \;\; = R_z(\alpha)\; R_y(-\beta)\; R_z(\gamma)$$

dus $\quad D(-\pi,0,0)\;\; D(\alpha,\beta,\gamma)\;\; D(0,0,\pi) = D(\alpha,-\beta,\gamma)$

$$\sum_{\mu\mu'} D_{m'\mu}(-\pi,0,0)\;\; D_{\mu\mu'}(\alpha,\beta,\gamma)\;\; D_{\mu',m}(0,0,\pi) = D_{m'm}(\alpha,-\beta,\gamma)$$

$$\sum_{\mu\mu'} e^{im'\pi}\, \delta_{m'\mu}\;\; D_{\mu\mu'}(\alpha,\beta,\gamma)\;\; \delta_{\mu',m}\, e^{-im\pi} = D_{m'm}(\alpha,-\beta,\gamma)$$

$$(-1)^{m'-m}\;\; D_{m'm}(\alpha,\beta,\gamma) = D_{m'm}(\alpha,-\beta,\gamma)$$

$$(-1)^{m'-m}\, d_{m'm}(\beta) = d_{m'm}(-\beta) = d_{mm'}(\beta)$$

$$d_{m'm}(\beta) = (-1)^{m'-m}\, d_{mm'}(\beta) \qquad \text{q.e.d.}$$

Summary

$$d^j_{m'm}(\pi-\beta) = (-1)^{j+m'}\, d_{m,-m'}(\beta) \qquad = (-1)^{j+m'}\, \cancel{(-1)^{m+m'}\, d_{-m',m}}\;\; d_{m',-m}(\beta)$$

$$\downarrow$$

$$d^j_{m'm}(\pi+\beta) = (-1)^{j+m'}\, d_{-m',m}(\beta)$$

$$(-1)^{j+2m'-m}\; d_{-m',m}(\beta) = (-1)^{j+m}$$

$$d^j_{m'm}(\beta) = (-1)^{m'-m}\, d_{-m,-m}(\beta)$$

$$d^j_{m'm}(\beta) = (-1)^{m'-m}\, d_{m,m'}(\beta) \qquad \underline{\text{omwisselen}}$$

inversie:

$$Y_{l,m}(\pi-\theta,\varphi+\pi) = (-1)^{l}\, Y_{l,m}(\theta,\varphi)$$

$$Y_{lm}(\pi-\theta,\varphi) = (-1)^{l+\cancel{m}}\, Y_{l,m}(\theta,\varphi)$$

$$Y_{lm}(-\theta,\varphi) = (-1)^{\cancel{m}}\, Y_{l,m}(\theta,\varphi)$$

$$d^j_{m'm}(\beta) = \cancel{d^j_{m,m'}\ldots}\;\; d^j_{-m,-m'}(\beta)$$

$$d^j_{m'm}(-\beta) = d^j_{mm'}(\beta) = (-1)^{m-m'}\, d_{m'm}(\beta) = d_{-m',-m}(\beta)$$