27/8/2000

(1)

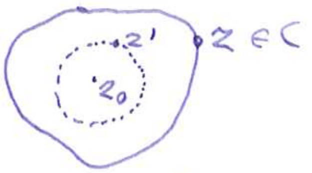
Analytic function $f(z) = f(z_0) + (z-z_0) f'(z_0) + O(|z|^2)$

Cauchy theorem $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$

Thus: the values $f(z)$ on the contour determines the function inside C

Use this to find the expansion around z_0

Choose z' such that $|z'-z_0| < |z-z_0|$ with z on C



$$f(z') = \frac{1}{2\pi i} \oint \frac{f(z)}{z-z'} dz$$

$$\frac{1}{z-z'} = \frac{1}{z-z_0-(z'-z_0)} = \frac{1}{z-z_0} \frac{1}{1-\frac{z'-z_0}{z-z_0}}$$

Let $S_n = 1 + x + \dots + x^n$

$xS_n = \quad x + \dots + x^n + x^{n+1}$

$(1-x) S_n = 1 - x^{n+1} \qquad x \neq 1 \qquad S_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$

$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \frac{x^{n+1}}{1-x}$

$|x|<1 \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$|x|>1 \qquad \frac{1}{1-x} = -\frac{1}{x} \cdot \frac{1}{1-\frac{1}{x}} = -\frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n = -\sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^n$

$$\frac{1}{z-z'} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{z'-z_0}{z-z_0}\right)^n = \sum_{n=0}^{\infty} \frac{(z'-z_0)^n}{(z-z_0)^{n+1}}$$

$$f(z') = \frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} \frac{(z'-z_0)^n}{(z-z_0)^{n+1}} f(z)\, dz$$

$$f(z') = \sum_{n=0}^{\infty} (z'-z_0)^n \underbrace{\frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz}_{= \frac{1}{n!} f^{(n)}(z_0)}$$

~~[illegible]~~

Check: $f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z)$

$$\frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz = \sum_{n=0} \frac{1}{2\pi i} \oint \frac{(z-z_0)^n}{n!} f^{(n)}(z) \cdot \frac{1}{(z-z_0)^{n+1}} dz$$

$$= \sum_{n=0} f^{(n)}(z_0) \underbrace{\frac{1}{2\pi i} \oint \frac{1}{z-z_0} dz}_{1} = f^{(n)}(z_0)$$

Hermition operator determines spectral decomposition of Hilbert space

$$H_0 |E_i^0\alpha\rangle = E_i^0 |E_i^0\alpha\rangle$$

Projection operators $\quad \hat P_i^0 = \sum_\alpha |E_i^0\alpha\rangle\langle E_i^0\alpha|$

$$\left.\begin{array}{l}\hat P_i^0 \hat P_i^0 = \hat P_i^0 \\ \hat P_i^0 \hat P_j^0 = 0 \quad i\neq j\end{array}\right\} \quad \hat P_i^0 \hat P_j^0 = \delta_{ij}\hat P_i^0 \; ; \quad [\hat P_i^0, \hat P_j^0] = 0$$

$$\hat H_0 = \sum_{i\alpha} |E_i^0\alpha\rangle E_i^0 \langle E_i^0\alpha| = \sum_i E_i^0 \hat P_i^0$$

$$[\hat H_0, \hat P_i^0] = 0 \; ; \quad Q_i^0 \equiv 1 - P_i^0 \; ; \quad P_i^0 + Q_i^0 = 1 \quad \cancel{[P_i^0, Q_j^0] \neq 0}$$

$$P_i^0 Q_i^0 = P_i^0(1-P_i^0) = P_i^0 - (P_i^0)^2 = P_i^0 - P_i^0 = 0$$

$$P_i^0 Q_j^0 = P_i^0(1-P_j^0) = P_i^0 - P_i^0 P_j^0 = P_i^0$$

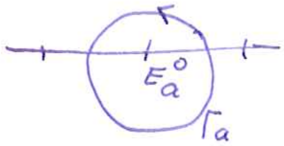
$$[P_i^0, Q_j^0] = 0$$

$$(\hat H_0)^n = \Big(\sum_i E_i^0 P_i^0\Big)^n = \sum_{i_1 i_2 \dots i_n} E_{i_1}^0 E_{i_2}^0 \cdots E_{i_n}^0 \underbrace{P_{i_1}^0 P_{i_2}^0 \cdots P_{i_n}^0}_{= P_{i_1}\delta_{i_1 i_2}\delta_{i_1 i_2}}$$

$$= \sum_i (E_i^0)^n P_i^0$$

$$f(\hat H_0) = \sum_i f(E_i^0) P_i^0$$

Resolvent ($\forall_i \; z \neq E_i^0$) $\quad G_0(z) \equiv (z - H_0)^{-1} = \sum_i \frac{P_i^0}{z - E_i^0}$



$$\frac{1}{2\pi i}\oint_{\Gamma_a} G_0(z) = \sum_i P_i^0 \underbrace{\frac{1}{2\pi i}\oint_{\Gamma_a} \frac{1}{z - E_i^0}\, dz}_{\delta_{a,i}} = P_a^0$$

$$P_a^0 = \frac{1}{2\pi i}\oint_{\Gamma_a} G_0(z)\, dz$$

Series expansions of $G_0(z)$ around $z - E_0^a$

→ formally Laurent series

only negative power $(z - E_0^a)^{-1}$ is one term in $G_0(z)$

Define reduced Resolvent $R_a^{(0)}(z) \equiv Q_a^{(0)} G_0(z) Q_a^{(0)} = \sum_{i \neq a} \frac{P_i^{(0)}}{z - E_i^{(0)}}$

$$G_0(z) = \frac{P_a^0}{z - E_a^0} + R_a^{(0)}(z)$$

Reduced resolvent $R_a^{(0)}$ maybe expanded in Taylor series around E_a^0

choose contour around E_a^0 inside of the singularities

$$R_a(z') = \frac{1}{2\pi i}\oint_{\Gamma_a} \frac{R(z)}{z' - z}\, dz$$

[just as before: $(z_0 \to E_a^0)$ 27/8/2000 ③

$$R_a(z') = \sum_{n=0}^{\infty} (z' - E_a^0)^n \frac{1}{2\pi i} \oint_{\Gamma_a} \frac{R(z)}{(z - E_a^0)^{n+1}} \, dz$$

RHS

$$\frac{1}{2\pi i} \oint_{\Gamma_a} \frac{1}{(z - E_a^0)^{n+1}} \sum_{i \neq a} \frac{P_i}{z - E_i^0}$$

$$\hat{H}_a \equiv Q_a \hat{H}_0 Q_a = \sum_{i \neq a} E_i^0 P_i^0$$

$$R_a(z') = \frac{1}{z' - \hat{H}_a} = \frac{1}{z' - E_a^0 - (\hat{H}_a - E_a^0)} = \frac{-1}{(H_a - E_a^0)} \cdot \frac{1}{1 - \left(\frac{z - E_a^0}{H_a - E_a^0}\right)}$$

$$= \frac{1}{E_a^0 - H_a} \cdot \left[1 + \frac{z - E_a^0}{\hat{H}_a - E_a^0} + \left(\frac{z - E_a^0}{\hat{H}_a - E_a^0}\right)^2 + \dots \right]$$

$$= R_a(E_a^0) \sum_{n=0}^{\infty} (-1)^n (z - E_a^0)^n \left(\frac{1}{E_a^0 - H_a}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (z - E_a^0)^n R_a^{n+1}(E_a^0)$$

Thus Laurent series:

$$G_0(z) = \frac{P_a^0}{z - E_a^0} + \sum_{n=0}^{\infty} (-1)^n R_a^{n+1}(E_a^0)(z - E_a^0)^n$$

$$R_a^n(E_a^0) = \sum_{i \neq a} \frac{P_a^0}{(E_a^0 - E_i)^n}$$

$$H \equiv H_0 + V$$

$$G(z) \equiv \frac{1}{z - (H_0 + V)}$$

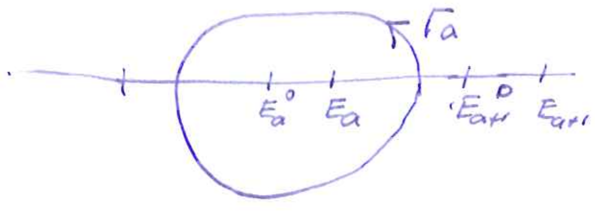
expand for small V

$$G(z) = \frac{1}{z - H_0 - V} = \frac{1}{z - H_0} \cdot \frac{1}{1 - (z - H_0)^{-1} V} = G_0(z) \frac{1}{1 - G_0(z) V}$$

$$G(z) = G_0(z) \left[1 + G_0 V + (G_0 V)^2 + \dots \right]$$

$$G(z) = \sum_{n=0}^{\infty} G_0(z) [V G_0(z)]^n$$

choose contour around E_a^0 as well as E_a



$$P_a = \frac{1}{2\pi i}\oint_{\Gamma_a} G(z)\,dz = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\oint G_0(z)\,(V G_0(z))^n\,dz$$

$$P_a = P_a^0 + \sum_{n=1}^{\infty} A_a^{(n)}$$

$$A_a^{(n)} = \frac{1}{2\pi i}\oint G_0(z)\,[V G_0(z)]^n\,dz$$

To substitute expansion of $G_0(z)$ introduce notation:

$$G_0(z) = \frac{P_a^0}{z-E_a^0} + \sum_{n=1}^{\infty} (-1)^{n-1} R_a^n(E_a^0)\,(z-E_a)^{n-1}$$

$$\left.\begin{aligned} S_a^0 &\equiv -P_a^0 \\ n>0 \quad S_a^{(n)} &\equiv R_a^n(E_a^0) \end{aligned}\right\} \quad G_0(z) = \sum_{k=0}^{\infty} (-1)^{k-1} S_a^{(k)}\,(z-E_a^0)^{k-1}$$

$$A_a^{(n)} = \frac{1}{2\pi i}\oint \underset{(0)}{G_0(z)}\,V\,\underset{(1)}{G_0(z)}\,V \ldots V\,\underset{(n)}{G_0(z)}\,dz$$

$$= \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} (-1)^{k_0+k_1+\ldots+k_n-(n+1)} \frac{1}{2\pi i}\oint_{\Gamma_a} (z-E_a^0)^{k_0+k_1+\ldots+k_n-(n+1)}\,dz$$

$$\times\; S_a^{(k_0)}\,V\,S_a^{(k_1)}\,V \ldots V\,S_a^{(k_n)}$$

$$\sum_{i=0}^{n} k_i = n \;\Rightarrow\; \frac{1}{2\pi i}\oint_{\Gamma_a} (z-E_a^0)^{-1}\,dz = 1$$

$$A_a^{(n)} = -\sum_{k_0+k_1+\ldots+k_n=n} S_a^{(k_0)}\,V\,S_a^{(k_1)}\,V \cdots V\,S_a^{(k_n)}$$

$n=1$ $\quad A_a^{(1)} = -S_a^{(0)} V S_a^{(1)} - S_a^{(1)} V S_a^{(0)} = P_a^0 V R_a^{(1)}(E_a^0) + R_a^{(1)}(E_a^0) V P_a^0$

$$A^{(1)} = PVR + RVP$$

$$A^{(2)} = PVPVR^2 + PVR^2VP + R^2VPVP$$
$$- PVRVR - RVPVR - RVRVP$$

Expansion of HP

$$G(z) = \frac{1}{z-\hat{H}} = \sum_i \frac{P_i}{z-E_i}$$

$$\left.\begin{aligned} &\frac{1}{2\pi i}\oint_{\Gamma_a} z\,G(z)\,dz = \frac{1}{2\pi i}\oint \frac{z P_a}{z-E_a}\,dz \\ &\frac{1}{2\pi i}\oint \frac{z}{z-E_a}\,dz = E_a \end{aligned}\right\} = E_a P_a = \hat{H} P_a$$

$$\begin{aligned} HP_a &= \frac{1}{2\pi i}\oint_{\Gamma_a} z\,G(z)\,dz \\ &= \sum_n \frac{1}{2\pi i}\oint_{\Gamma_a} z\,G_0(VG_0)^n\,dz \\ &= \frac{1}{2\pi i}\oint_{\Gamma_a} z\,G_0\,dz + \sum_{n=1}^{\infty}\frac{1}{2\pi i}\oint_{\Gamma_a} z\,G_0(VG_0)^n\,dz \\ &= E_a^0 P_a^0 + \sum_{n=1}^{\infty}\frac{1}{2\pi i}\oint_{\Gamma_a} z\,G_0(VG_0)^n\,dz \end{aligned}$$

$$z = z - E_a^0 + E_a$$

$$\begin{aligned} HP_a &= E_a^0 P_a^0 + E_a^0 \sum_{n=1}^{\infty}\frac{1}{2\pi i}\oint_{\Gamma_a} G_0(VG_0)^n\,dz + \sum_{n=1}^{\infty}\frac{1}{2\pi i}\oint (z-E_a^0)\,G_0(VG_0)^n\,dz \\ &= E_a^0 \cdot P_a + \sum_{n=1}^{\infty} B_a^{(n)} \end{aligned}$$

$$B_a^{(n)} = \frac{1}{2\pi i}\oint (z-E_a^0)\,G_0(VG_0)^n\,dz$$

Like $A_a^{(n)}$ except $\frac{1}{2\pi i}\oint_{\Gamma_a} (z-E_a^0)\,(z-E_a^0)^{k_0+\dots+k_n-(n+1)}\,dz$

$$k_0 + \dots + k_n = n-1$$

$$(-1)^{k_0+\dots k_n-(n+1)} = (-1)^{n-1-(n+1)} = 1$$

$$\left\{\begin{aligned} B_a^{(n)} &= \sum_{k_0+k_1+\dots+k_n=n-1} S_a^{(k_0)} V S_a^{(k_1)} V \dots V S_a^{(k_n)} \\ (H-E_a^0)\,P_a &= \sum_{n=1}^{\infty} B_a^{(n)} \\ P_a &= P_a^0 + \sum_{n=1}^{\infty} A_a^{(n)} \\ A_a^{(n)} &= -\sum_{k_0+k_1+\dots k_n=n} S_a^{(k_0)} V S_a^{(k_1)} V \dots V S_a^{(k_n)} \end{aligned}\right.$$

Non degenerate

eigenvector $P_a | E_a^0 \rangle$

Norm $\langle E_a^0 | P_a P_a | E_a^0 \rangle = \langle E_a^0 | P_a | E_a^0 \rangle$

eigenvalue $H P_a = E_a P_a$

$tr(H P_a) = E_a \, tr(P_a) = E_a$

$$E_a = tr(H P_a) = E_a^0 + \sum_{n=1}^{\infty} tr(B_a^{(n)})$$

$$B_a^{(1)} = P_a^0 V P_a^0$$

$$B_a^{(2)} = P_a^0 V P_a^0 V R_a' + PVRVP + RVPVP$$

$$B_a^{(3)} = -PVPVPVR^2 + PVPVR^2VP - PVR^2VPVP + R^2VPVPVP + PVRVRVP$$

$tr(B^{(n)}) \Rightarrow \langle 0 | B^{(n)} | 0 \rangle \rightarrow R$ only inside

$$\varepsilon^{(1)} = \langle 0 | V | 0 \rangle$$

$$\varepsilon^{(2)} = \langle 0 | VRV | 0 \rangle$$

$$\varepsilon^{(3)} = \langle 0 | VRVRV | 0 \rangle - 2 \langle 0 | V R^2 V | 0 \rangle \langle 0 | V | 0 \rangle$$